Final Exam
1/5/06, 1PM-4PM

Your Name: _______________________________________________

ID Number: _______________________________________________

Closed book. One sheet of notes (double sided) permitted. Write your solutions in the space provided next to the questions (use both sides if necessary). If you need extra space, you are probably on the wrong track!

<table>
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<th>PROBLEM NO.</th>
<th>FULL POINTS</th>
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<td>Total Points</td>
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1. (10 pt) Color representation
(a) (2pt) Suppose you have three separate projectors that can project red, green, and blue lights with different intensities. What will be the perceived color of the projected screen if all three projectors project lights at the same intensity? What will be the color if the red and green projectors project lights at the same intensity, and the blue projector is not projecting any light?

(b) (2pt) What will be the color of the paint if you mix equal amount of cyan, magenta, and yellow paints? What will be the color if you mix equal amount of magenta and yellow paints?

(c) (3pt) Suppose a particular color has the following R,G, B values: R=100, G=200, B=50. What will be the Y, I, Q values if you convert it to YIQ representation?

Hint: The RGB <-> YIQ conversion formula is given below.

\[
\begin{bmatrix}
Y \\
I \\
Q \\
\end{bmatrix} = 
\begin{bmatrix}
0.299 & 0.587 & 0.114 \\
0.596 & -0.274 & -0.322 \\
0.211 & -0.523 & 0.311 \\
\end{bmatrix} \begin{bmatrix}
R \\
G \\
B \\
\end{bmatrix},
\begin{bmatrix}
Y \\
I \\
Q \\
\end{bmatrix} = 
\begin{bmatrix}
1.0 & 0.956 & 0.621 \\
1.0 & -0.272 & -0.649 \\
1.0 & -1.106 & 1.703 \\
\end{bmatrix} \begin{bmatrix}
Y \\
I \\
Q \\
\end{bmatrix}
\]

(d) (3pt) Suppose you give a presentation using a LCD projector, and you notice that the color tone of the projected slides does not look right. It seems to be too yellowish. What do you think may be the problem?
2. (15pt) Consider a 2D image \( f(x,y) = \cos(6\pi x + 4\pi y) \).
   
   a) (3pt) Determine its Fourier transform \( F(u,v) \) and illustrate the spectrum (i.e., the impulses in the transform) in Fig. (a).
   
   b) (3pt) Suppose this signal is sampled uniformly with sampling intervals \( \Delta x = \Delta y = \Delta = 1/5 \). Draw the spectrum of the sampled signal in Fig. (b).
   
   c) (4 pt) Suppose the sampled signal is interpolated by an ideal low-pass filter \( h_1(x,y) \) with frequency response

   \[
   H_1(u,v) = \begin{cases} 
   \Delta^2 & \frac{f_s}{2} < u, v < \frac{f_s}{2}, \\
   0 & \text{otherwise}
   \end{cases}
   \]

   where \( f_s = 1/\Delta \). Draw the spectrum of the reconstructed signal in Fig. (c). Give the spatial representation of the reconstructed signal \( f_r1(x,y) \).
   
   d) (5pt) Suppose that the sample-and-hold filter (also called nearest neighbor) is used instead. Its filter response can be written as

   \[
   h_2(x,y) = h(x)h(y),
   \]

   where

   \[
   h(x) = \begin{cases} 
   1 & \text{if } |x| \leq \Delta/2 \\
   0 & \text{otherwise}
   \end{cases}
   \]

   Give the filter response in the frequency domain, \( H_2(u,v) \). If the filter is further band-limited to \(-7 \leq u,v \leq 7\), give the frequency and spatial representation of the reconstructed signal \( f_{r2}(x,y) \).
Fig. (a): Original Signal

Fig. (b): Sampled Signal

Fig. (c): Reconstructed Signal
3. (15 pt) Consider a source with 3 symbols \{a, b, c\} with the following probability distribution:

Marginal probabilities:
\[ p(a) = \frac{1}{2}, \quad p(b) = \frac{1}{3}, \quad p(c) = \frac{1}{6} \]

Conditional probabilities:
\begin{align*}
    p(a|a) &= \frac{2}{3}, \quad p(b|a) = \frac{1}{6}, \quad p(c|a) = \frac{1}{6}; \\
    p(a|b) &= \frac{1}{3}, \quad p(b|b) = \frac{1}{3}, \quad p(c|b) = \frac{1}{3}; \\
    p(a|c) &= \frac{1}{2}, \quad p(b|c) = \frac{1}{4}, \quad p(c|c) = \frac{1}{4}
\end{align*}

(a) (5 pt) Design a Huffman codebook for coding one symbol at a time. Calculate the average bit rate per symbol. Compare it to first order entropy.

(b) (10 pt) Design a Huffman codebook for coding two symbols at a time. Calculate the average bit rate per symbol. Compare it to second order entropy.
For your convenience, the probably distributions are reproduced here:
Marginal probabilities:
\[ p(a) = \frac{1}{2}, \ p(b) = \frac{1}{3}, \ p(c) = \frac{1}{6} \]

Conditional probabilities:
\[ p(a/a) = \frac{2}{3}, \ p(b/a) = \frac{1}{6}, \ p(c/a) = \frac{1}{6}; \]
\[ p(a/b) = \frac{1}{3}, \ p(b/b) = \frac{1}{3}, \ p(c/b) = \frac{1}{3}; \]
\[ p(a/c) = \frac{1}{2}, \ p(b/c) = \frac{1}{4}, \ p(c/c) = \frac{1}{4} \]
4. (10 pt) Consider the following predictive coding scheme. The pixels in an image are scanned from left to right and from top to bottom. Each new pixel is predicted by the average of the pixel to the left and top. Let \( f \) and \( \hat{f} \) represent the original and the predicted values, and \( e = f - \hat{f} \) the prediction error. The prediction error is quantized according to:

\[
\hat{e} = \begin{cases} 
B, & e \geq 0 \\
-B, & e < 0 
\end{cases}
\]

For the image given in Fig. (a) and use \( B=3 \) as an example, write the predicted image, the error image, the quantized error image, and the reconstructed image in Figs. (b), (c), (d) and (e), respectively. Note that you should use the reconstructed value in the previous position for prediction. For simplicity, skip the processing for the pixels in the first row and column. Assuming these pixels are coded directly and losslessly. **In order to save the time, you only need to complete the processing for the second row.**

![Image of the processed images](image_url)
5. (10 pt) You are given one image $f_1$. You are asked to synthesize an image $f_2$ that is $f_1$ shifted 10 pixels to the right and then rotated by 30 degree in counter-clock direction.
(a) Write down the forward mapping function from $f_1$ to $f_2$, and the inverse mapping function from $f_2$ to $f_1$. Assuming $f_1$ uses image coordinate $(u,v)$, and $f_2$ uses image coordinate $(x,y)$.
(b) What mapping function you would use to generate $f_2$?
6. (20 pt) Write MATLAB functions (or C program) to perform following:
(a) For a given input image, down-sample it by a factor of 2 in both horizontal and vertical directions. Use the 2x2 averaging filter for pre-filtering.
(b) For a given input image, up-sample it by a factor of 2 in both horizontal and vertical directions. Use bilinear interpolation.

You should assume the input image has only the luminance component, and is saved in an unsigned character array. Your output image (for both parts) should also be saved in an unsigned character array. (i.e., perform necessary truncation)

Your function should have the following syntax:
[outimg]=down2(inimg); [outimg]=up2(inimg)
7. (20pt) Write a MATLAB script for implementing a simplified transform coder. (pseudo JPEG!) Your program should do the following for each image block of 8x8:
1) shift the image mean value to 0 by subtracting all pixel values by 128;
2) perform 8x8 DCT;
3) quantize the DCT coefficients using a given quantization matrix Qmatrix, scaled by a
given factor QP; For simplicity, assume all the transform coefficients have a dynamic
range represented by (-M,M), where M is a known value;
4) count the number of non-zero coefficients after quantization, return this number;
5) perform inverse DCT on the quantized coefficients;
6) shift the mean value back by adding 128 to all reconstructed pixel values;
7) perform necessary truncation so that the reconstructed pixel values can be represented
by 8 bit unsigned integers, and save the results in the output image;
8) determine the MSE of the reconstructed block and return this value;

Finally, you should sum the numbers of non-zero coefficients for all blocks, and
determine the ratio defined as total_number_nonzero_coefficients/total_number_pixels as
an approximate compression ratio. Also determine the average MSE of all blocks, and
convert it to the average PSNR value. Your program should report the estimated
compression ratio and the PSNR, as well as the decompressed image.

For simplicity, assuming the image has only the luminance component represented in 8
bit unsigned integer and it has been read into an unsigned character array inimg[][]. Your
matlab function should have the following syntax:

[outimg, compression_ratio, PSNR]=DCT_coding(inimg,QP,Qmatrix)

Note that you can use built-in functions in MATLAB for computing 2D DCT and inverse
DCT. Also, you are encouraged to use the “blkproc” function of MATLAB to speed up
the processing. But this is not required.
8. (10 pt, Bonus) In linear prediction, we predict the value of a sample $f_0$ by its previous $K$ samples $f_1, f_2, ..., f_K$, using $f_{0,p} = \sum_{k=1}^{K} a_k f_k$.

We derived the solution for the optimal linear predictor that will minimize the mean square error of linear prediction in the class. Notice that with this solution, the prediction coefficients may not sum to 1. In order to maintain the mean value of the original pixels, it is desirable to design a predictor where the prediction coefficients sum to 1 (i.e., $\sum_{k=1}^{K} a_k = 1$). Derive the optimal predictor under this constraint. You can either derive the general solution for an arbitrary value of $K$, or consider a special case of $K=3$, if it is easier for you to solve this special case.

(hint: you could set $a_K = 1 - \sum_{k=1}^{K-1} a_k$, and determine the coefficients $a_1, a_2, ..., a_{K-1}$ that will minimize the mean square prediction error).