Final Exam
12/17/2008, 3PM-5:50PM

Your Name: _______________________________________________
ID Number: _______________________________________________

Closed book. One sheet of notes (double sided) permitted. For problems 1 and 2, write your solutions in the space provided next to the questions (use both sides if necessary). For other problems, use the provided bluebook.

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<th>PROBLEM NO.</th>
<th>FULL POINTS</th>
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1. (20pt) (a) (10pt) Find the opening of the binary image, F, in Fig. (a) by the structuring element H in Fig. (b). You can use the grids in Figs. (c) to Fig. (e) to draw the intermediate and the final results.

(b) (10 pt) Find the gray scale closing of the function sketched in Fig. (f) with the structural element in Fig. (g). Sketch the intermediate and final result in Figs. (h) to Fig. (j).

Comment on the effect of these operations.
2. (20pt) Consider a 2D image \( f(x,y) = \cos(4\pi x) \sin(8\pi y) \).

a) (5pt) Determine its Fourier transform \( F(u,v) \) and illustrate the spectrum (i.e., the impulses in the transform) in Fig. (a).

b) (5pt) Suppose this signal is sampled uniformly with sampling intervals \( \Delta x = \Delta y = \Delta = 1/6 \). Draw the spectrum of the sampled signal in Fig. (b).

c) (5pt) Suppose the sampled signal is interpolated by an ideal low-pass filter \( h_1(x,y) \) with

\[
H_1(u,v) = \begin{cases} 
\Delta^2 & -f_s/2 < u, v < f_s/2, \\
0 & \text{otherwise}
\end{cases}
\]

where \( f_s = 1/\Delta \). Draw the spectrum of the reconstructed signal in Fig. (c). Give the spatial representation of the reconstructed signal \( f_{r1}(x,y) \).

d) (5pt) Suppose that the interpolation filter has frequency response \( H_2(u,v) = H_x(u) H_y(v) \), with \( H_x(u) \) shown below. Draw the spectrum of the reconstructed signal in Fig. (d). Give the frequency and spatial representation of the reconstructed signal \( f_{r2}(x,y) \).

Note: if you have problem deriving the spectrum of the given \( f(x,y) \), for parts (b)-(d), assume \( f(x,y) = \cos(4\pi x + 8\pi y) \).
Fig. (a) Spectrum of original signal

Fig. (b) Spectrum of sampled signal

Fig. (c) Spectrum of interpolated signal by $H_1(u,v)$. Write down the spatial domain representation $f_{r1}(x,y)$ below.

Fig. (d) Spectrum of interpolated signal by $H_2(u,v)$. Write down the spatial domain representation $f_{r2}(x,y)$ below.
3. (10 pt) Consider a source with 3 symbols \{a, b, c\} with the following probability distribution:

Marginal probabilities: \( p(a)=1/2, \ p(b)=3/8, \ p(c)=1/8 \)

Conditional probabilities: \( p(a/a)=2/3, p(b/a)=1/6, p(c/a)=1/6; \)
\( p(a/b)=1/3, p(b/b)=1/3, p(c/b)=1/3; \)
\( p(a/c)=1/2, p(b/c)=1/4, p(c/c)=1/4 \)

(a) (5 pt) Design a Huffman codebook for coding one symbol at a time. Calculate the average bit rate per symbol. Compare it to first order entropy.

(b) (5 pt) Design a conditional Huffman codebook for coding a new symbol given its previous symbol is “a”. Calculate the average bit rate per symbol. Compare it the conditional entropy for this condition.

4. (10 pt) You are given two 1-D basis vectors
\[
\begin{bmatrix}
1 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
-1
\end{bmatrix}, \begin{bmatrix}
1 \\
1
\end{bmatrix}, \begin{bmatrix}
-1 \\
1
\end{bmatrix}
\]

(a) Create 4 2-D basis images using the above basis vectors;

(b) Given an image \( F = \begin{bmatrix} 6 & 5 \\ 3 & 2 \end{bmatrix} \), determine the transform coefficients, \( t_k, k=1,2,3,4 \).

(c) Quantize the coefficients using a uniform quantizer with a step size of 2. Using a quantizer that is centered at 0 (i.e. there is a reconstruction level at 0).

(d) Determine the reconstructed image using quantized coefficients.

5. (10 pt) Consider a 2x2 transform with basis vectors
\[
\begin{bmatrix}
\cos \theta \\
\sin \theta
\end{bmatrix}, \begin{bmatrix}
-\sin \theta \\
\cos \theta
\end{bmatrix}
\]

(a) (3pt) Prove that they form an orthonormal basis for any arbitrary \( \theta \).

(b) (3pt) If the original random vector \( f \) has a covariance matrix of
\[
\begin{bmatrix}
1 & \rho \\
\rho & 1
\end{bmatrix}
\]
where \(|\rho|<1\). What is the covariance matrix of the transformed vector \( t \)?

(c) (4pt) If we want to design the transform basis so that the transformed coefficients are uncorrelated, what should be the value of \( \theta \)?

6. (10 pt) You are given one image \( f_1 \). You are asked to synthesize an image \( f_2 \) that is \( f_1 \) shifted 5 pixels to the right, 2 pixels up, and then rotated by 30 degree in clockwise direction.

(a) Write down the forward mapping function from \( f_1 \) to \( f_2 \), and the inverse mapping function from \( f_2 \) to \( f_1 \). Assuming \( f_1 \) uses image coordinate \((u,v)\), and \( f_2 \) uses image coordinate \((x,y)\).

(b) What mapping function you would use to generate \( f_2 \)?
7. (20 pt) (a) Write a MATLAB function to create a 3-level Gaussian pyramid representation. For
down sampling, you should use a 3x3 averaging filter for pre-filtering. Your samples should start
with the top row and top column. For pixels on the boundary (top and bottom row, leftmost and
rightmost column), you don’t need to perform filtering.

Your program should have a syntax of

\[ [\text{GP1}, \text{GP2}, \text{GP3}] = \text{GaussianPyramid}(\text{inimg}); \]

where GP1 is the bottom level image (same resolution as inimg), and GP2 is the middle level (half
size in both dimension), and GP3 is the top level image (quarter size in both dimension).

You should create a function that does filtering and downsampling, and call this function from your
main function.

(b) Write a MATLAB function to create a 3-level Laplacian pyramid, given the approximation
pyramid created in part (a). For interpolation, use the bilinear filter.

Your program should have a syntax of

\[ [\text{LP1}, \text{LP2}, \text{LP3}] = \text{LaplacianPyramid}(\text{GP1}, \text{GP2}, \text{GP3}); \]

where GP1, GP2, GP3 are as defined before, LP1, LP2, LP3 are the bottom, middle, top level images in
the Laplacian Pyramid.

You should create a function that performs bilinear interpolation, and call this function from your
main function.

You should assume the input image has only the luminance component, and is saved in an unsigned
character array. Your output image (for both parts) can be saved as floating point numbers.