Review: Fourier Transforms and Sampling Theorem

1 Review of Fourier Transforms

It is expected that the student is knowledgeable in the basic theory of signals and systems. Background in discrete-time signal and systems can be found in Chapters 1, 2, and 4 of the *Schaum’s outline* of Digital Signal Processing by Monson H. Hayes (ISBN 0-07-027389-8).

It is expected that the student is comfortable with both continuous-time and discrete-time signal and systems. Only a brief summary of the definitions is given here, except for the discrete Fourier transform (DFT) which will be covered in greater detail.

1.1 Continuous-Time Fourier Transforms

**Impulse Functions**

The impulse satisfies the following properties

\[ \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \]

\[ \int_{-\infty}^{\infty} f(t) \delta(t) \, dt = f(0) \]

provided \( f(t) \) is continuous at \( t = 0 \).

\[ \int_{-\infty}^{\infty} f(t) \delta(t - t_o) \, dt = f(t_o) \]

provided \( f(t) \) is continuous at \( t = t_o \).

It follows that the impulse is the *convolutional identity*:

\[ f(t) \ast \delta(t) = f(t) \]

\[ f(t) \ast \delta(t - t_o) = f(t - t_o) \]

**Fourier Series**

The Fourier series of a \( T \)-periodic signal \( x(t) \) is

\[ C(k) = \frac{1}{T} \int_{T} x(t) e^{-jk\omega_0 t} \, dt \]

where \( \omega_0 = 2\pi/T \).

\[ x(t) = \sum_{k} C(k) e^{jk\omega_0 t} \]
Fourier Transform

\[ X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \]

\[ x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega \]

It is a good exercise to try to derive the properties of the Fourier transform by yourself.

An important Fourier transform pair concerns the impulse function:

\[ \mathcal{F}\{\delta(t)\} = 1 \]

and

\[ \mathcal{F}^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \]

The following example is very important for developing the sampling theorem.

**Example**

1. Derive the Fourier series of the \( T \)-periodic impulse-train

\[ \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

*Answer*: By direct calculation using properties of impulse functions one obtains the Fourier series coefficients

\[ P_k = \frac{1}{T} \]

Therefore, one has

\[ \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{k} e^{jk\omega_0 t} \quad \omega_0 = \frac{2\pi}{T} \]

2. Derive the Fourier transform \( X(\omega) \) of a general \( T \)-periodic signal in terms of its Fourier series coefficients \( C(k) \).

*Answer*: Write

\[ x(t) = \sum_{k} C(k) e^{jk\omega_0 t} \]
Then the Fourier transform is given by

\[ X(\omega) = \mathcal{F}\left\{ \sum_k C(k) e^{j k \omega_0 t} \right\} \]

\[ = \sum_k C(k) \mathcal{F}\left\{ e^{j k \omega_0 t} \right\} \]

\[ = \sum_k C(k) (2\pi \delta(\omega - k \omega_0)) \]

\[ = 2\pi \sum_k C(k) \delta(\omega - k \omega_0) \]

which is simply a \textit{weighted} impulse train.

3. Derive the Fourier transform \( \Delta_T(\omega) \) of the \( T \)-periodic impulse-train \( \delta_T(t) \).

\textit{Answer}: For the \( T \)-periodic impulse train \( \delta_T(t) \), we have the Fourier series coefficients \( \Delta_T(k) = \frac{1}{T} \). Therefore

\[ \Delta_T(\omega) = \frac{2\pi}{T} \sum_k \delta(\omega - k \omega_0) \]

where \( \omega_0 = \frac{2\pi}{T} \), or

\[ \Delta_T(\omega) = \omega_0 \sum_k \delta(\omega - k \omega_0) \]

That is, the Fourier transform of a periodic impulse train is again a periodic impulse train. This is a key ingredient of the sampling theorem, a cornerstone of signal processing.

1.2 Discrete-Time Fourier Transform

Discrete-Time Fourier Transform (DTFT)

\[ X^f(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \]

\[ x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^f(\omega) e^{j\omega n} d\omega \]

\( X^f(\omega) \) is the Discrete-time Fourier transform (DTFT). Note that by definition \( X^f(\omega) \) always periodic with period \( 2\pi \).

Discrete Fourier Transform (DFT)

\[ X^d(k) = \sum_{n=0}^{N-1} x(n) W_N^{-nk} \]

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where $W_N = e^{j2\pi/N}$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X^d(k) W_N^{nk}$$

More detail on the DFT will be given below.

## 2 Sampling Theorem

The sampling theorem states that if $x(t)$ is bandlimited to $(-\omega_o, \omega_o)$, then it can be reconstructed from its samples $x(nT)$ provided that the sampling frequency $\omega_s = 2\pi/T$ satisfies

$$\omega_s \geq 2\omega_o$$

The sampling theorem can be derived using the impulse train considered earlier.

Ideal sampling can be written as a multiplication of the signal $x(t)$ by the periodic impulse train.

$$x_s(t) = x(t) \cdot \delta_T(t)$$

There are two useful ways to write the Fourier transform of $x_s(t)$. The first way is direct, the second way uses the convolution theorem. First way:

$$x_s(t) = x(t) \cdot \delta_T(t)$$

$$= x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(t) \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)$$

So ...

$$X_s(\omega) = \mathcal{F}\{x_s(t)\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \mathcal{F}\{\delta(t - nT)\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT}$$

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Second way:

\[
X_s(\omega) = \frac{1}{2\pi} X(\omega) * \Delta_T(\omega)
\]

\[
= \frac{1}{2\pi} X(\omega) * \omega_o \sum_k \delta(\omega - k\omega_o)
\]

\[
= \frac{1}{T} \sum_k X(\omega) * \delta(\omega - k\omega_o)
\]

\[
= \frac{1}{T} \sum_k X(\omega - k\omega_o)
\]

If \(\omega_s - \omega_o \geq \omega_o\), then there will no overlap between adjacent copies of the spectrum of \(x(t)\) and it can be recovered using a lowpass filter. It can be seen from either representation that \(X_s(\omega)\) is periodic with period \(2\pi/T\).

What is the Fourier transform of the discrete-time signal \(x_s(n) := x(nT)\)? Note that the discrete-time signal is different from \(x_s(t)\). The signal \(x_s(t)\) is an impulse train, and the area of the impulses are equal to the samples values of \(x(t)\). \(x_s(n)\) on the other hand is a true discrete-time signal. To obtain the DTFT \(X_s^f(\omega)\) begin with the definition

\[
X_s^f(\omega) = \sum_n x_s(n) e^{-j\omega n}
\]

\[
= \sum_n x(nT) e^{-j\omega n}
\]

and compare the two representations above for \(X_s(\omega)\) to get

\[
X_s^f(\omega) = X_s\left(\frac{\omega}{T}\right).
\]

**Reconstruction from Samples**

It can be seen by the Fourier transform that the original signal \(x(t)\) can be recovered from its samples \(x(nT)\) by passing \(x_s(t)\) through an ideal lowpass filter having the frequency response

\[
H(\omega) = \begin{cases} 
T & |\omega| < \omega_c \\
0 & |\omega| > \omega_c
\end{cases}
\]

where the cut-off frequency \(\omega_c\) satisfies

\[
\omega_o < \omega_c < \omega_s - \omega_o
\]
The output $y(t)$ of the filter can be written as
\[ Y(\omega) = H(\omega) \cdot X_s(\omega) \]
\[ y(t) = h(t) \ast x_s(t) \]
\[ y(t) = h(t) \ast \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) \]
\[ y(t) = \sum_{n=-\infty}^{\infty} x(nT) [h(t) \ast \delta(t - nT)] \]
\[ y(t) = \sum_{n=-\infty}^{\infty} x(nT) h(t - nT) \]
where
\[ h(t) = F^{-1}\{H(\omega)\} \]
\[ = \text{sinc}(t/T_c) \]
where $T_c = 2\pi/\omega_c$.

**Remarks** The ideal sampling can not be physically achieved exactly, nor can the ideal lowpass filter. The sampling of a signal is made inexact by the fact that (1) the signal will not be exactly band limited and (2) the samples must be quantized. The ideal lowpass filter can not be realized because it is not causal (in fact it is infinitely supported in both directions so even if a delay is acceptable there is no shift to make it causal). In addition, the ideal lowpass filter is not even BIBO stable — the integral of $|h(t)|$ is not finite.

In practice, signals are lowpass filtered before sampling is done — in order to ensure the signal is appropriately bandlimited before the signal is sampled. This lowpass filter that comes before the sampling is performed is called an anti-aliasing filter because its purpose is to avoid the spectral overlap, or aliasing, that can occur when the sampling rate is not sufficiently high. The loss of information due to the anti-aliasing filter is much preferable to the loss of information due to aliasing.

In addition, the sampling rates used in practice are slightly higher than the sampling theorem demands, because the lowpass anti-aliasing filter will not be an ideal lowpass filter. The higher sampling rate also reduces the error due to the quantization of signal samples.

The text describes various circuits that are used to implement sampling. Usually two steps are required: a hold circuit that produces a step-like analog signal followed by the sampling circuit.