EL 7133: Digital Signal Processing  
Instructor: Ivan Selesnick

Final Exam

• 6 single-sided pages of notes are allowed. Otherwise the test is closed notes.
• Closed book.
• Show your work!
• Simplify your answers.
• Calculators are allowed.

1. Prove that the frequency response of a discrete-time LTI system, 
   \[ H(e^{j\omega}) = \sum_n h(n) e^{-jn\omega}, \]
   where \( h(n) \) is the impulse response, is a 2\(\pi \) periodic function of \( \omega \).

2. DFT. Find the DFT of the \( N \)-point discrete-time signal \( x(n) \),
   \[ x(n) = [1, 1, \ldots, 1, 0, 0, \ldots, 0], \quad n = 0, 1, \ldots, N - 1. \]
   The \( N \)-point signal consists of \( M \) 1’s followed by \((N - M)\) 0’s. Your answer should be a simplified formula for \( X(k) \) in terms of \( k, M, \) and \( N \). If \( X(k) \) is real, write it without using \( j \). If \( X(k) \) is complex but the phase is linear, write it as a linear-phase term times a real-valued expression.

3. The transfer function \( H(z) \) of an FIR filter has 13 zeros in the \( z \)-plane as illustrated.

   ![Diagram of zeros on the unit circle](image)

   The zeros on the unit circle are at powers of \( W_{12} \). The dc-gain of the filter is unity. All the poles are at \( z = 0 \).
   (a) Accurately sketch the impulse response of the filter.
   (b) Sketch the frequency response magnitude \( |H(e^{j\omega})| \). Accurately indicate the nulls of the frequency response.
   (c) Sketch the phase of the frequency response \( \angle H(e^{j\omega}) \).

4. Rate changing. A discrete-time signal \( x(n) \) having a rate of 24 samples per second must be converted into a new discrete-time signal \( y(n) \) having a rate of 18 samples per second.
   (a) Sketch a multirate system that performs the necessary sampling rate conversion. Sketch the frequency responses of the required filter(s).
   (b) Because ideal ‘brick-wall’ filters can not be realized, how would you proceed with the design of filters(s) for this multirate system?

5. Scaling. Suppose the signal \( x(n) \) is bounded by 0.5, meaning that \( |x(n)| < 0.5 \) for all \( n \). The impulse response of an FIR digital filter is
   \[ h(n) = \{1.5, 2.0, -0.5, 0.3, 0, 0, \ldots \} \]
   To ensure that the output signal \( y(n) \) does not exceed 1 in absolute value, how should \( x(n) \) be scaled before filtering? Show your work.
6. **Power spectral densities.** The figures below show the power spectral density (PSD) of four stationary discrete-time random processes. The following figures also show a realization (signal) generated using each of the four PSDs, but they are out of order. Match each signal to its most likely PSD by completing the table.

<table>
<thead>
<tr>
<th>PSD</th>
<th>Signal</th>
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7. **Spectrograms.** Consider the spectrogram of the following signal.

![Spectrogram](image)

The spectrograms were computed with parameters:
- \( R \in \{20, 80\} \)
- \( L \in \{2, 25\} \)
- \( N \in \{20, 256\} \)

where
- \( R \) = block length
- \( L \) = time lapse between segments.
- \( N \) = FFT length (Each signal block is zero-padded to length \( N \).)

Indicate values \( R \), \( L \), and \( N \) by completing the table.

<table>
<thead>
<tr>
<th>Spectrogram</th>
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<th>( L )</th>
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<tbody>
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8. **Half-band filter design.** Design a Type I linear-phase FIR digital half-band filter of minimal length having a transfer function \( H(z) \) of the form

\[
H(z) = Q(z) (1 + z^{-1})^2 (1 + z^{-1} + z^{-2})
\]

(a) Find the impulse response \( h(n) \) of the filter.
(b) Sketch the zeros of \( H(z) \) in the complex \( z \)-plane.
(c) Roughly sketch the frequency response \( |H(e^{j\omega})| \) based on the filter’s zero-diagram and half-band property.
(d) What particular properties does the filter have, when it is used for interpolation?

9. **Fractional-delay filters.** Consider the ideal discrete-time fractional-delay system for a delay of a half-sample, \( \tau = 0.5 \).

(a) Sketch \( |H(e^{j\omega})| \) for \(-2\pi \leq \omega \leq 2\pi \).
(b) Sketch the phase response, \( \angle H(e^{j\omega}) \), for \(-2\pi \leq \omega \leq 2\pi \).
(c) Roughly sketch the impulse response \( h(n) \) for \(-3 \leq n \leq 4 \).

10. **AR modeling.** Consider a stationary random signal generated by the causal difference equation

\[
x(n) = d(n) - a_1 x(n-1) - a_2 x(n-2)
\]

where the driving signal \( d(n) \) is a stationary white noise signal with variance \( \sigma^2 \).

![X(n) Diagram](image)

Find the values of \( a_1 \), \( a_2 \), and \( \sigma^2 \) so that the autocorrelation function \( r_x(k) \) of the generated random signal \( x(n) \) has the values:

\[
r_x(0) = 2, \quad r_x(1) = 1, \quad r_x(2) = 0.
\]