The Characterization and Design of Hilbert Transform Pairs of Wavelet Bases

Ivan W. Selesnick
Electrical Engineering, Polytechnic University
6 Metrotech Center
Brooklyn, NY 11201
selesi@aco.poly.edu

Abstract — Several authors have demonstrated that dramatic improvements can be obtained in wavelet-based signal processing by utilizing a pair of wavelet transforms where the wavelets form a Hilbert transform pair. But how should one design the two low-pass scaling filters so that the two wavelets form a Hilbert transform pair? In this paper we derive the simple condition the two lowpass filter must satisfy — they must be offset from one another by a half sample. The derivation is based on the limit functions defined by the infinite product formula. This gives an alternative derivation and explanation for the result by Kingsbury, that the dual-tree DWT is (nearly) shift-invariant when the scaling filters satisfy the same offset.

I Introduction

Hilbert transforms pairs of wavelets bases are very useful for signal processing for several reasons, including the following.

1. The implementation of a nearly shift invariant DWT with limited redundancy.

2. The implementation of complex wavelet transforms where the spectrum of the complex wavelet is single sided. For natural images the correlation between the magnitude of the complex coefficient in adjacent scales is stronger than for real DWTs, which substantially improves the performance of denoising algorithms (for example) that exploit the correlation between adjacent scales [3].

3. The efficient implementation of two-dimensional directional DWTs based on separable filter banks.

Work by Kingsbury [7, 8] on the dual-tree DWT, by Freeman et al [5, 10] on steerable pyramids, and by Abry and Flandrin [1, 2] on transient detection, clearly demonstrate the advantages of signal processing methods that call for two wavelet transforms, where one wavelet is (approximately) the Hilbert transform of the other.

Kingsbury’s dual-tree DWT is a wavelet transform based simply on two parallel independent critically sampled DWTs, as in Fig. 1. However, the filters in each DWT are design to work together for maximum benefit. On one hand, the dual-tree DWT can be viewed as an overcomplete wavelet transform with a redundancy factor of two. On the other hand, the dual-tree DWT is also a complex DWT, where the first and second DWTs represent the real and imaginary parts of a single complex DWT.

Note that a wavelet is a band-pass signal; however, it is fully determined by the filters h0(n), g0(n), neither of which is band-pass. How can we choose the lowpass filters h0 and g0 so that the two wavelets form a Hilbert transform pair? This is the question addressed in this paper.

Kingsbury found that the dual-tree DWT is nearly shift-invariant when the lowpass filters of one DWT interpolate midway between the lowpass filters of the second DWT. This paper considers the limit functions defined by the infinite-product formula, rather than the (near) shift-invariance of a finitely iterated filter bank as in [7], and arrives at the same condition. This paper thereby gives an alternative explanation for why the scaling filters should be designed to be offset from each other by a half sample delay.

A Preliminaries

Let the filters h0(n), h1(n) represent a CQF pair. That is,

$$\sum_n h_0(n) h_0(n + 2k) = \delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

and h1(n) = (-1)^(1-n) h0(1-n). Equivalently, in terms of the Z-transform, we have

$$H_0(z)H_0(1/z) + H_0(-z)H_0(-1/z) = 2$$

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and

\[ H_1^v(z) = \frac{1}{z} H_0^v(-1/z). \]

We use the notation \( H^v(z) \) for the \( z \)-transform of \( h(n) \); then the frequency response of the filter is \( H(\omega) = H^v(e^{j\omega}) \). The filters \( g_0(n), g_1(n) \) represent another CQF pair. In this paper we assume \( h(n), g(n) \) are real-valued filters. The dilation and wavelet equations give the scaling and wavelet functions,

\[ \phi_h(t) = \sqrt{2} \sum_n h_0(n) \phi_h(2t - n), \]
\[ \psi_h(t) = \sqrt{2} \sum_n h_1(n) \phi_h(2t - n). \]

The scaling function \( \phi_g(t) \) and wavelet \( \psi_g(t) \) are defined similarly, but with filters \( g_0(n) \) and \( g_1(n) \).

II Hilbert transform pairs

Recall the definition of the Hilbert transform. \( \psi_h(t) \) is the Hilbert transform of \( \psi_h(t), \psi_g(t) = H \{ \psi_h(t) \} \), if

\[ \Psi_g(\omega) = \begin{cases} -j \Psi_h(\omega), & \omega > 0 \\ j \Psi_h(\omega), & \omega < 0 \end{cases} \]

where

\[ \Psi_h(\omega) = \mathcal{F} \{ \psi_h(t) \}, \]
\[ \Psi_g(\omega) = \mathcal{F} \{ \psi_g(t) \}. \]

The frequency response of the lowpass filters are given by \( H_0(\omega) \) and \( G_0(\omega) \),

\[ H_0(\omega) = \text{DTFT} \{ h_0(n) \}, \]
\[ G_0(\omega) = \text{DTFT} \{ g_0(n) \}. \]

If \( \psi_h(t) \) and \( \psi_g(t) \) form Hilbert transform pair, then

\[ |\Psi_h(\omega)| = |\Psi_g(\omega)| \]

and therefore we should have

\[ |H_0(\omega)| = |G_0(\omega)|. \]

That is, the two lowpass filters are related as follows.

\[ G_0(\omega) = H_0(\omega) e^{-j \theta(\omega)} \]

where \( \theta(\omega) \) is 2\( \pi \)-periodic. Now the question is: how should we choose the phase function \( \theta(\omega) \) so that the two wavelets generated by \( h_0 \) and \( g_0 \) form a Hilbert transform pair? We proceed by considering 3 questions.

A How is \( \phi_g(t) \) related to \( \phi_h(t) \)?

By the infinite-product formula we have

\[ \Phi_h(\omega) = \mathcal{F} \{ \phi_h(t) \} = \Phi_h(0) \prod_{k=1}^{\infty} \left\{ \frac{1}{\sqrt{2}} H_0 \left( \frac{\omega}{2k} \right) \right\}. \]

Similarly for \( \Phi_g(\omega) \),

\[ \Phi_g(\omega) = \mathcal{F} \{ \phi_g(t) \} = \Phi_g(0) \prod_{k=1}^{\infty} \left\{ \frac{1}{\sqrt{2}} G_0(\frac{\omega}{2k}) \right\} \]

B How is \( G_1(\omega) \) related to \( H_1(\omega) \)?

The CQF filter bank has

\[ H_1^v(z) = \frac{1}{z} H_0^v(-1/z) \quad \text{or} \quad H_1(\omega) = e^{-j\omega} H_0(\omega - \pi). \]

Similarly,

\[ G_1(\omega) = e^{-j\omega} G_0(\omega - \pi) \]
\[ = e^{-j\omega} H_0(\omega - \pi) e^{-j(\omega(\omega - \pi))} \]
\[ = e^{-j\omega} H_0(\omega - \pi) e^{j(\omega - \pi)} \]
\[ = H_1(\omega) e^{j(\omega - \pi)}. \]

C How is \( \psi_g(t) \) related to \( \psi_h(t) \)?

The Fourier transform of \( \psi_h(t) \) is given by

\[ \Psi_h(\omega) = \mathcal{F} \{ \psi_h(t) \} = \frac{1}{\sqrt{2}} H_1 \left( \frac{\omega}{2} \right) \Phi_h \left( \frac{\omega}{2} \right) \]

and similarly for \( \Psi_g(\omega) \),

\[ \Psi_g(\omega) = \mathcal{F} \{ \psi_g(t) \} = \frac{1}{\sqrt{2}} G_1 \left( \frac{\omega}{2} \right) \Phi_g \left( \frac{\omega}{2} \right) \]

Therefore we can write

\[ \Psi_g(\omega) = \Psi_h(\omega) e^{j[\theta(\omega/2 - \pi) - \sum_{n=2}^{\infty} \theta(\omega/2^n)]}. \]
D Phase condition

Can we choose \( \theta(\omega) \) so that \( \psi_h(t), \psi_g(t) \) make a Hilbert transform pair? From (3) and (1) we see that \( \theta(\omega) \) must satisfy the following condition.

\[
\theta(\omega/2 - \pi) - \sum_{k=2}^{\infty} \theta(\omega/2^k) = \begin{cases} 
-\frac{\pi}{2}, & \omega > 0 \\
\frac{\pi}{2}, & \omega < 0.
\end{cases}
\]  

(4)

It can be shown that if the 2\( \pi \)-periodic function \( \theta(\omega) \) is defined as

\[
\theta(\omega) = \frac{\omega}{2}, \quad |\omega| < \pi
\]

(5)
as illustrated in Fig. 2, then condition (4) holds and \( \psi_h(t), \psi_g(t) \) make a Hilbert transform pair. First, note that if \( \theta(\omega) \) is given by (5), then \( \theta(\omega/2 - \pi) \) is a 4\( \pi \)-periodic function given by

\[
\theta(\omega/2 - \pi) = \begin{cases} 
-\frac{\pi}{2} + \frac{\pi}{4}, & 0 < \omega < 2\pi \\
\frac{\pi}{2} - \frac{\pi}{4}, & -2\pi < \omega < 0
\end{cases}
\]
as illustrated in Fig. 2. Now if we call the second term in (4) \( \beta(\omega) \)

\[
\beta(\omega) = -\sum_{k=2}^{\infty} \theta(\omega/2^k)
\]

and if \( \theta(\omega) \) is given by (5), then we can show that \( \beta(\omega) \) is given by

\[
\beta(\omega) = \begin{cases} 
-\frac{\pi}{2}, & |\omega| < 4\pi \\
\beta(\omega - 4\pi), & 4\pi < \omega < 4\pi + 2\pi \\
\beta(\omega + 4\pi), & \omega < -4\pi
\end{cases}
\]
as illustrated in Fig. 2. Adding \( \theta(\omega/2 - \pi) \) and \( \beta(\omega) \), we get the graph shown in Fig. 2. Evidently, one finds that condition (4) is indeed satisfied by the choice (5).

Theorem: If \( H_0(\omega) \) and \( G_0(\omega) \) are lowpass CQF filters (scaling filters) with

\[
G_0(\omega) = H_0(\omega) e^{-j\pi/2} \quad \text{for} \quad |\omega| < \pi,
\]
then the corresponding wavelets are a Hilbert transform pair,

\[
\psi_g(t) = H(\psi_h(t)).
\]

Equivalently, the digital filter \( g_0(n) \) is a half-sample delayed version of \( h_0(n) \),

\[
g_0(n) = h_0(n-1/2).
\]

As a half-sample delay can not be implemented with an FIR filter (not even a rational IIR filter can be exact), it is necessary to make an approximation.

III Design Problem

The phase condition leads to following design problem: Construct the shortest FIR filters \( h_0, g_0 \) such that they possess a specified number of zero moments, and that

\[
G_0(\omega) \approx H_0(\omega) e^{-j\pi/2}.
\]

The error function is given by

\[
E_1(\omega) = G_0(\omega) - H_0(\omega) e^{-j\pi/2}.
\]

If we define a new function \( E_3(\omega) \) by

\[
E_3(\omega) := E_1(2\omega) = G_0(2\omega) - H_0(2\omega) e^{-j\omega}
\]
then the Z-transform \( E_3(z) \) is a polynomial,

\[
E_3(z) = G_0(z^2) - H_0(z^2) z^{-1}.
\]

Let us choose \( z = 1 \) as the point of approximation as it is the middle of the passband of the lowpass filter. To make \( E_3(z) \) close to zero at \( z = 1 \), we can ask that

\[
G_0(z^2) - H_0(z^2) z^{-1} = Q_1(z) (1 - z^{-1})^k
\]
where \( L \) represents the degree of approximation to the half-sample delay. If \( K \) is the number of zero wavelet moments, and \( L \) is the parameter for controlling the half-sample delay approximation, then we have the following design equations, which we wish to solve for the filters \( h_0 \) and \( g_0 \) of minimal length.

1. \( \sum_n h_0(n) h_0(n + 2k) = \delta(k) \)
2. \( \sum_n g_0(n) g_0(n + 2k) = \delta(k) \)
3. \( H_0(z) = Q_1(z) (1 + z^{-1})^K \)
4. \( G_0(z) = Q_2(z) (1 + z^{-1})^K \)
5. \( G_0(z^2) - z^{-1} H_0(z^2) = Q_1(z) (1 - z^{-1})^k \)

We illustrate two examples obtained using this design problem. The design equations are nonlinear; however, solutions can be obtained using Gröbner bases [4]. We used the software Singular [6] to obtain the Gröbner bases needed for the following examples.

Example 1: With \( K = 4 \) and \( L = 5 \), we find that the shortest filters \( h_0(n) \) and \( g_0(n) \) satisfying the conditions are of length
10. Fig. 3 illustrates one of the several solutions that exist. Note that \([H_0(\omega)]\) and \([G_0(\omega)]\) are indistinguishable in the plot. The phase of the function of \(G(\omega)/H(\omega)\), denoted \(\theta(\omega)\), shows its agreement with \(\omega/2 \approx 0\). The plot of the function \(\psi_h(\omega) + j \psi_g(\omega)\) shows that it approximates zero for \(\omega < 0\) as expected if \(\psi_h\) and \(\psi_g\) make a Hilbert transform pair.

**Example 2:** With \(K = 3\) and \(L = 7\), the minimal lengths of \(h_0(n)\) and \(g_0(n)\) is again 10 samples. Fig. 4 illustrates one of the several solutions. It can be seen that the wavelets are not quite as smooth as in the previous example, but that \([\psi_h(\omega) + j \psi_g(\omega)]\) is closer to zero for negative frequencies. This is to be expected, as we have reduced the number of zero moments and at the same time increased the degree of approximation for the half-sample delay.

Using longer filters, we have obtained solutions that have both good smoothness and good half-sample delay properties. Note that \(h_0(n)\) and \(g_0(n)\) do not need to have (near) linear phase in order for \(\psi_h(t)\) and \(\psi_g(t)\) to make a Hilbert transform pair, although it may be desirable for other reasons depending on the application.

**IV Conclusion**

Using the infinite product formula, it was shown that for two orthogonal wavelets to form a Hilbert transform pair, the scaling filters should be offset by a half sample. An example was presented to illustrate the trade-off between the number of zero wavelet moments and the degree of half-sample delay approximation. The examples given in this paper, and other examples, are available on the authors' webpage, taco.polyp.edu/selesi/

In addition, we have a few other recent results concerning the design of Hilbert wavelet pairs (HWP). The design problem presented in this paper cannot be solved using the spectral factorization approach. Instead, we used Gröbner bases to solve the nonlinear design equations. However, we have also developed a simple design procedure based on the flat delay filter and spectral factorization. Using that method, we can also easily generate biorthogonal solutions that satisfy symmetry conditions, as well as IIR solutions, which can be efficient for real-time implementations.

Recently, we have also obtained Hilbert transforms pairs of wavelet frames where each of the two frames is generated using an oversampled filter bank of the type described in [9]. The resulting solutions have a higher degree of smoothness because there are more degrees of freedom, and an improved time-frequency bandwidth product.

**References**


Figure 3: Example of near Hilbert transform pair of orthonormal wavelet bases, with $N = 10$, $K = 4$, $L = 5$.

Figure 4: Example of near Hilbert transform pair of orthonormal wavelet bases, with $N = 10$, $K = 3$, $L = 7$. 