1. (10 pt) Assuming the probability density functions of a blood test result for patients with and without a disease are described by Gaussian distributions, where we assume \( \mu_n < \mu_d \):

\[
\text{Normal: } p_n(x) = \frac{1}{\sqrt{2\pi\sigma_n}} \exp\left\{-\left(\frac{x - \mu_n}{\sigma_n}\right)^2\right\};
\]
\[
\text{Diseased: } p_d(x) = \frac{1}{\sqrt{2\pi\sigma_d}} \exp\left\{-\left(\frac{x - \mu_d}{\sigma_d}\right)^2\right\};
\]
Assume the diagnosis is determined based on a threshold \( t \). For a patient with test value below \( t \), we call it normal. Otherwise, we call it diseased. Determine:

a) the sensitivity (true positive fraction) and
b) the specificity (true negative fraction) in terms of the threshold \( t \).

c) When you increase \( t \), how is the sensitivity and specificity affected?
d) Propose a method to choose \( t \) to achieve a good tradeoff between the sensitivity and specificity.

Note: you could just write down all the equations clearly without doing the actual integration.

2. (10 pt) (a) Sketch the typical shape of the X-ray energy spectrum. Assume the voltage applied between the anode and the cathode is 140 KV, and the binding energy of K-shell electrons in the tungsten anode is 60 KeV, and that of L-shell and M-shell are respectively 10 KeV and 5 KeV. (b) In X-ray imaging for medical applications, why are low-energy photons undesired? What measures can be taken to reduce the number of the low-energy photons entering the human body? (c) why are photons undergoing scattering undesired? What measures can be taken to prevent these photons from entering the detectors?

3. (25 pt) Consider the X-ray imaging of a medium consisting of two slabs separated by a distance of \( D \), as illustrated in Fig. 3A. Assume the slabs are infinitely long in the direction orthogonal to the plane shown. Assume each slab has a height of \( H \) and a thickness of \( T \). (a) (10pt) If the X-ray source is an ideal point source, illustrate the detected object extent along the detector axis (y-axis) on Fig. 3A in the problem sheet. What is the height of each slab on the detector plane? Write down the expression of the detected signal as a function of \( y \) and illustrate this function. Express your solution in term of given parameters \( I_0, D, D_1, D_2 \). In your solution, please assume the slab is very thin so that you can ignore the edge effect due to the thickness of the slab. (b) (10pt) Repeat (a) but assume the source is a small disk of diameter \( W=D \), and \( D_2=2D_1 \). Please illustrate the detected object extent on Fig. 3B in the problem sheet. What is the blurring width along the slab edges? Can you separate the two slabs? (c) (5pt) What is the equivalent impulse response of this X-ray imaging system when \( W=D \)? What is its resolution in terms of the minimal distance between two lines for the two lines to be separable? Based on your result, interpret your answer for part (b).
4. (20 pt) Suppose the tissue slice (with dimension 4cmx4cm) being imaged by a parallel beam x-ray CT scanner has a distribution of the linear attenuation coefficients as shown in Fig. 4, with \( \mu_1=2, \mu_2=3, \mu_3=4 \). (a) Assume the detector is a point detector. Sketch the projection \( g(l, \theta) \) as a function of \( l \), for \( \theta=0, 90 \) degrees, respectively. You should indicate the magnitudes of the projected values where necessary on your sketch. Also clearly specify any transition points in the \( l \)-axis. (b) Sketch the image obtained by backprojections from 0 and 90 degree projections, individually and together. (c) Determine the Fourier transform of the original image along a line with orientation \( \theta=90 \) degree in the frequency domain. (d) What will be the projected function for \( \theta=0 \) if the detector is an area detector with width 0.2 cm. Sketch the projection function.

Hint: you could make use of the following Fourier transform pair and property:

\[
g(x) = \begin{cases} 1, & \text{if } |x| < \Delta/2 \\ 0, & \text{otherwise} \end{cases} \leftrightarrow G(f) = \Delta \frac{\sin(\pi \Delta f)}{\pi \Delta f}
\]

\[
g(x - x_0) \leftrightarrow G(f) e^{-j2\pi fx_0}
\]

![Fig 4.](image)

5. (10 pt) Suppose the medium slice that you would like to image using parallel beam X-ray CT has a size of \( W \text{(cm)} \times W \text{(cm)} \) and a maximum spatial frequency of \( f_{\text{max}} \) (cycles/cm) in all directions. (a) What should be the detector spacing \( T \) to enable accurate image reconstruction? (b) How many detectors you need to have in the detector array (to accommodate projections in all directions)? (c) What would be the reconstructed image size (in terms of the number of pixels)? (d) Suppose you apply a windowing function to the rho-filter for reconstruction using either filtered backprojection or convolution backprojection. If the cutoff frequency \( \geq f_{\text{max}} \) and the detector is point detector, is the reconstructed image different from the original image? (e) What if the detector has a width equal to detector spacing \( T \)? How is the reconstructed image related to the original image?
6. (10 pt) Compare SPECT and PET: (a) For SPECT, what type of radioactive decay does the radio tracer undergo? Do you need collimator at the detector? Why? (b) Answer the same questions for PET. (c) How does the half-life of the radio tracer affect the imaging process? Should the imaging be done shortly after the radio tracer is given to the patient, or much later (more than the half life of the tracer?)

7. (15 pt) A 2-D slice to be imaged is shown below. The linear attenuation coefficients in different regions are \( \mu_1 = 1cm^{-1}, \mu_2 = 2cm^{-1}, \mu_3 = 3cm^{-1} \).

(a) Suppose a solution containing a gamma ray emitting radio tracer with an initial radioactivity of \( A_1 \) and half life \( T_1 \) fills section R1 and R2. We image the radioactivity distribution in this slice using a rotating SPECT camera at time \( t \) after the injection of the radio tracer. Compute the measured signal by the camera at positions A. Do you expect the reading at detector B to be different? What would be the detector readings at positions C and D, respectively?

(b) Now suppose the radio tracer in (a) is replaced by a positron emitting radio tracer with initial radioactivity of \( A_2 \) and half life \( T_2 \). This time the slice is imaged using a PET scanner. Compute the measured signal by the pair of detectors positioned at A and B, and at C and D.