

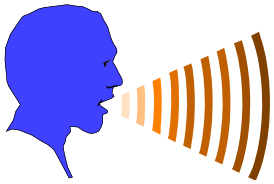
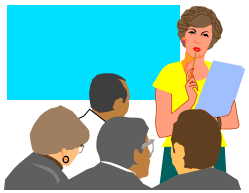



Image and Video Processing

Two-Dimensional Fourier Transform and Linear Filtering

Yao Wang
Polytechnic School of Engineering, New York University

Outline

- General concept of signals and transforms
  - Representation using basis functions
- Continuous Space Fourier Transform (CSFT)
  - 1D -> 2D
  - Concept of spatial frequency
- Discrete Space Fourier Transform (DSFT) and DFT
  - 1D -> 2D
- Continuous space convolution
- Discrete space convolution
- Convolution theorem

Signal in 2D Space

- General 2D continuous space signal: f(x,y)
  - Can have infinite support: x,y= (-infty,…, infty)
  - f(x,y) can generally take on complex values
- General 2D discrete space signal: f(m,n)
  - Can have infinite support: m,n= -infty,…, 0, 1,.., infty
  - f(m,n) can generally take on complex values
- Each color component of an image is a 2D real signal with finite support
  - MxN image: m=0,1,…,M-1, n=0,1,…,N-1
  - We will use first index for row, second index for column
  - We will consider a single color component only
  - Same operations can be applied to each component

Transform Representation of Signals

- Transforms are decompositions of a function f(x) into some basis functions Ø(x, u). u is typically the freq. index.

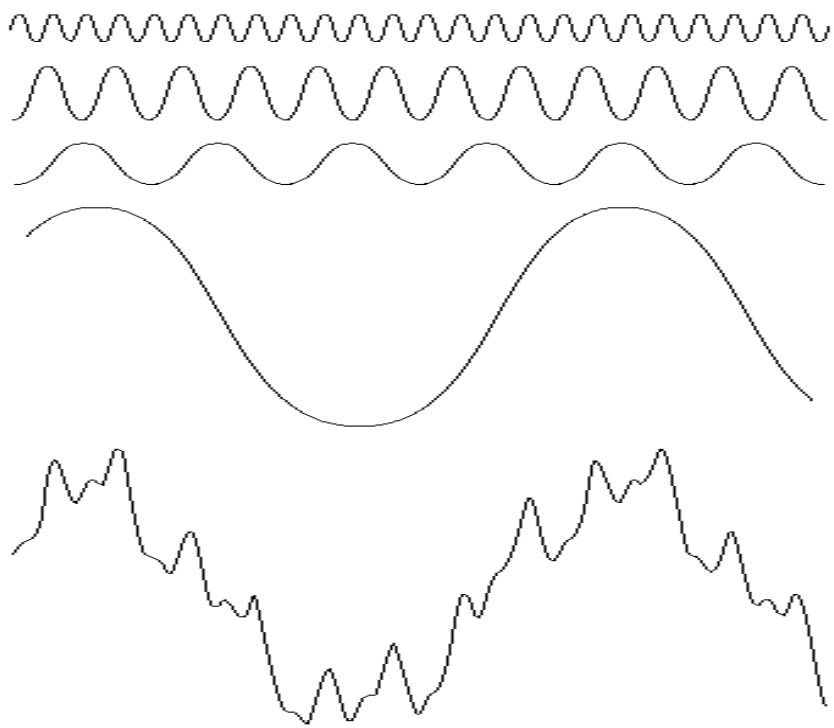
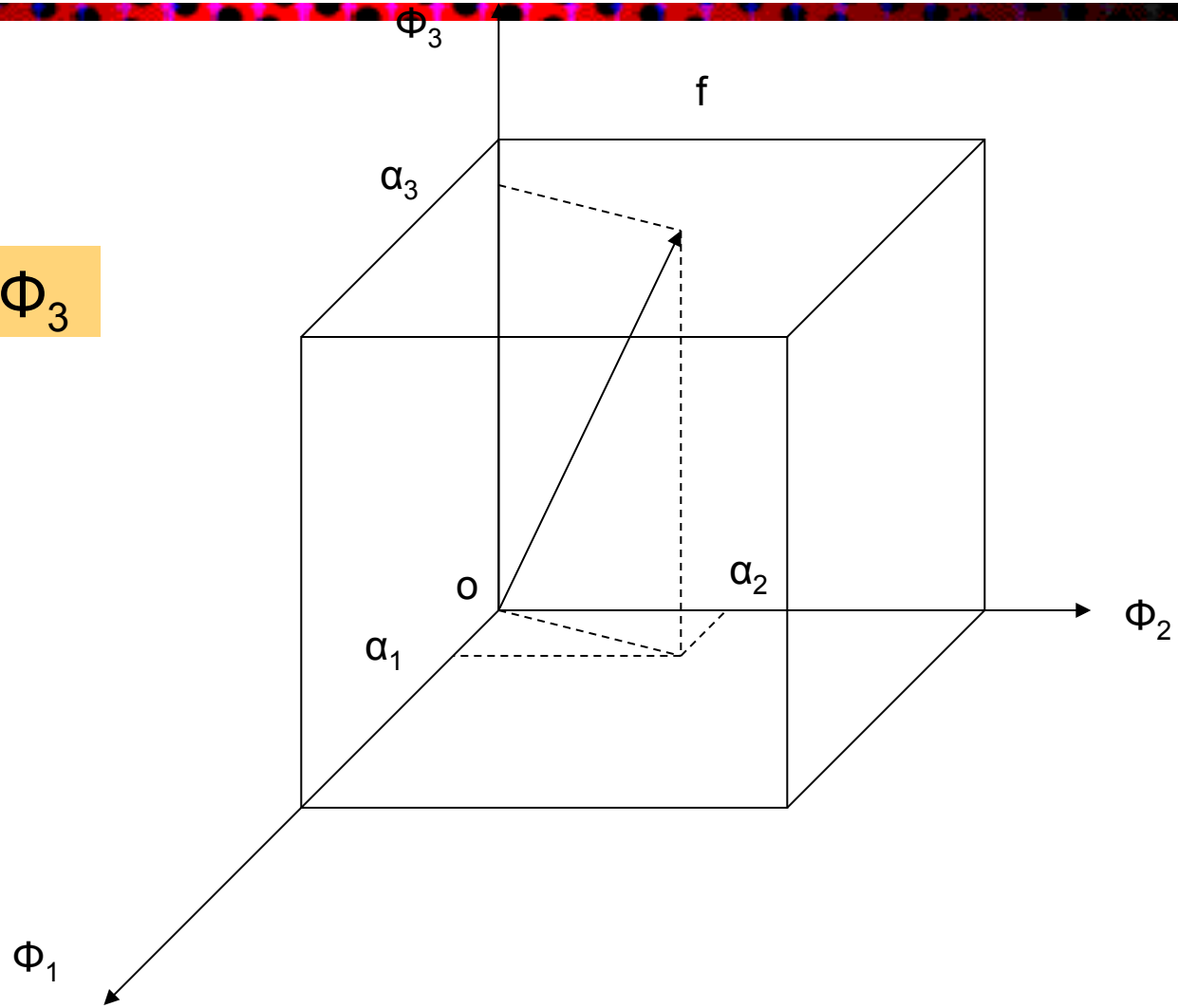


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Illustration of Decomposition in Vector Space

$$f = \alpha_1\Phi_1+\alpha_2\Phi_2+\alpha_3\Phi_3$$



Decomposition of 1D Signal

- Ortho-normal basis function

$$\int_{-\infty}^{\infty} \phi(x,u_1)\phi^*(x,u_2)dx = \begin{cases} 1, & u_1 = u_2 \\ 0, & u_1 \neq u_2 \end{cases}$$

- Forward transform

$$F(u) = < f(x), \phi(x,u) > = \int_{-\infty}^{\infty} f(x)\phi^*(x,u)dx$$

Projection of *f(x)* onto ϕ*(x,u)*

- Inverse transform

$$f(x) = \int_{-\infty}^{\infty} F(u)\phi(x,u)du$$

Representing *f(x)* as sum of *ϕ(x,u)* for all *u*, with weight *F(u)*

1D Continuous Time Fourier Transform

- Basis function

$$\phi(x,u) = e^{j2\pi ux}, \quad u \in (-\infty, +\infty).$$

- Forward Transform

$$F(u) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

- Inverse Transform

$$f(x) = F^{-1}\{F(u)\} = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

Important Transform Pairs

$$f(x)=1 \quad \Leftrightarrow \quad F(u)=\delta(u)$$

$$f(x)=e^{j2\pi f_0 x} \quad \Leftrightarrow \quad F(u)=\delta(u-f_0)$$

$$f(x)=\cos(2\pi f_0 x) \quad \Leftrightarrow \quad F(u)=\frac{1}{2}\left(\delta(u-f_0)+\delta(u+f_0)\right)$$

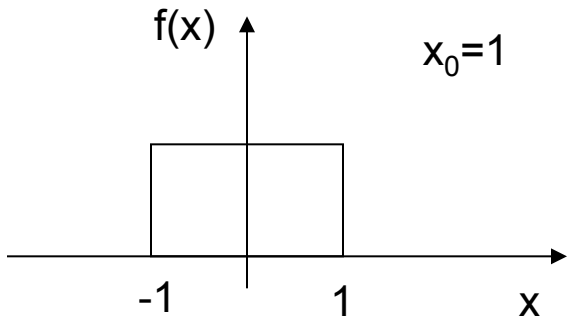
$$f(x)=\sin(2\pi f_0 x) \quad \Leftrightarrow \quad F(u)=\frac{1}{2j}\left(\delta(u-f_0)-\delta(u+f_0)\right)$$

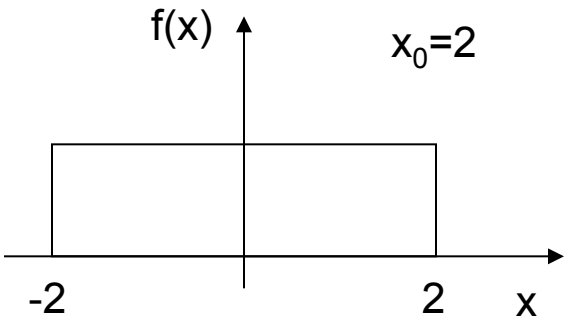
$$f(x)=\begin{cases} 1, & |x|<x_0 \\ 0, & otherwise \end{cases} \quad \Leftrightarrow \quad F(u)=\frac{\sin(2\pi x_0 u)}{\pi u}=2x_0\,\mathrm{sinc}(2x_0 u)$$

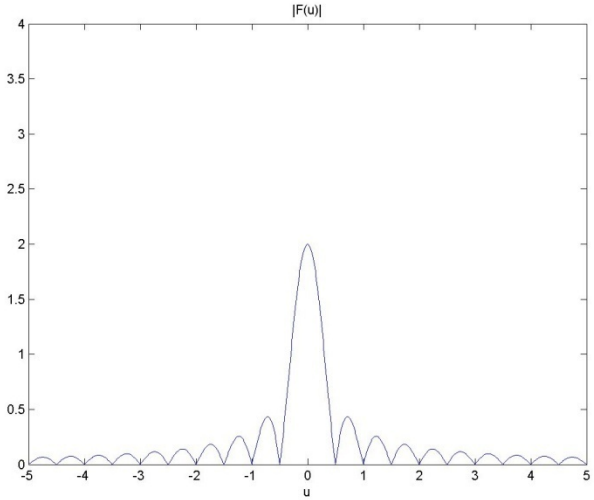
$$where, \mathrm{sinc}(t)=\frac{\sin(\pi t)}{\pi t}$$

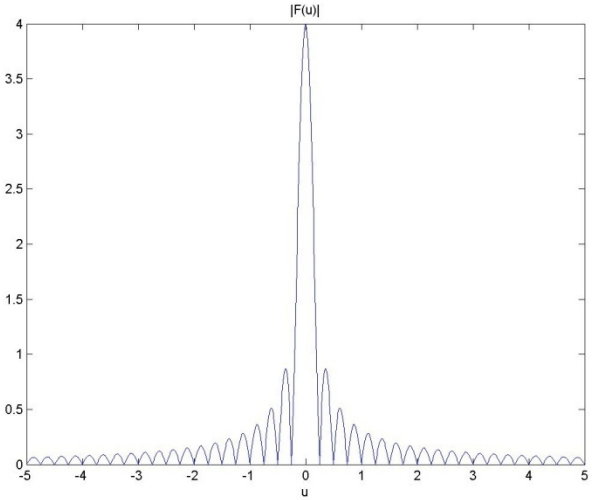
FT of the Rectangle Function

$$F(u) = \frac{\sin(2\pi x_0 u)}{\pi u} = 2x_0 \operatorname{sinc}(2x_0 u) \quad where, \operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$





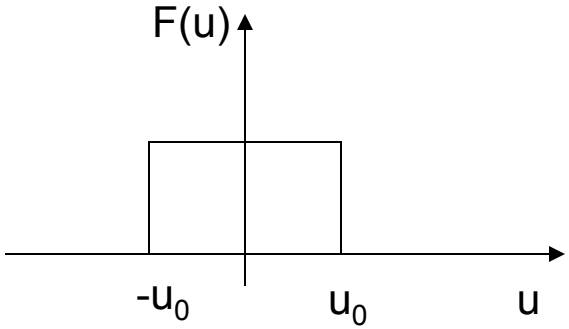




Note first zero occurs at u_0=1/(2 x_0)=1/pulse-width, other zeros are multiples of this.

IFT of Ideal Low Pass Signal

- What is f(x)?

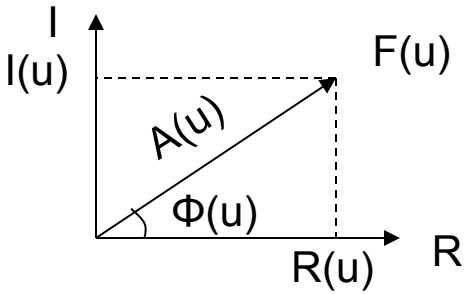


Representation of FT

- Generally, both f(x) and F(u) are complex
- Two representations
  - Real and Imaginary $F(u) = R(u) + jI(u)$
  - Magnitude and Phase

$$F(u) = A(u)e^{j\phi(u)}, \quad where$$

$$A(u) = \sqrt{R(u)^2 + I(u)^2}, \phi(u) = \tan^{-1}\frac{I(u)}{R(u)}$$



- Relationship

$$R(u) = A(u)\cos\phi(u), \quad I(u) = A(u)\sin\phi(u)$$

- Power spectrum

$$P(u) = A(u)^2 = F(u) \times F(u)^* = |F(u)|^2$$

What if f(x) is real?

- Real world signals f(x) are usually real
- F(u) is still complex, but has special properties

$$F^*(u) = F(-u)$$

$$R(u) = R(-u), A(u) = A(-u), P(u) = P(-u) : \text{even function}$$

$$I(u) = -I(-u), \phi(u) = -\phi(-u) : \text{odd function}$$

Property of Fourier Transform

- Duality

$$f(t) \Leftrightarrow F(u)$$
$$F(t) \Leftrightarrow f(-u)$$

- Linearity

$$F\{a_1 f_1(x) + a_2 f_2(x)\} = a_1 F\{f_1(x)\} + a_2 F\{f_2(x)\}$$

- Scaling

$$F\{af(x)\} = aF\{f(x)\}$$

- Translation

$$f(x - x_0) \Leftrightarrow F(u)e^{-j2\pi x_0 u}, \quad f(x)e^{j2\pi u_0 x} \Leftrightarrow F(u - u_0)$$

- Convolution

$$f(x) \otimes g(x) = \int f(x-\alpha)g(\alpha)d\alpha$$

$$f(x) \otimes g(x) \Leftrightarrow F(u)G(u)$$

We will review convolution later!

Two Dimension Continuous Space Fourier Transform (CSFT)

- Basis functions

$$\phi(x,y;u,v)=e^{j(2\pi ux+2\pi vy)}=e^{j2\pi ux}e^{j2\pi vy},\quad u,v\in(-\infty,+\infty).$$

- Forward – Transform

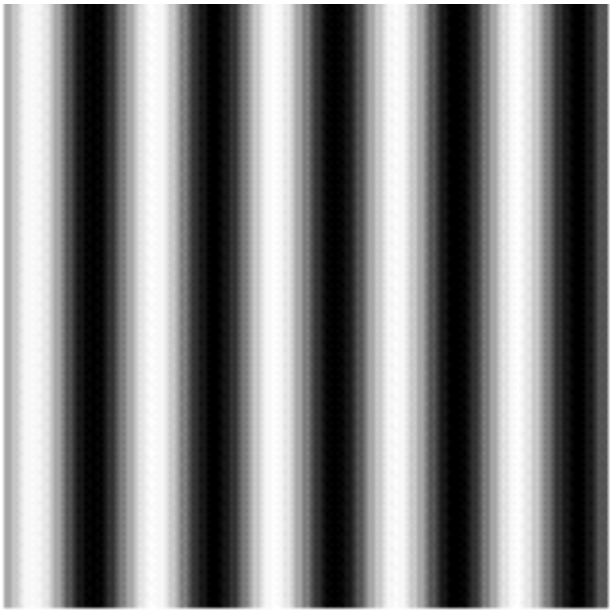
$$F(u,v)=F\{f(x,y)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

- Inverse – Transform

$$f(x,y)=F^{-1}\{F(u,v)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{j2\pi(ux+vy)}dudv$$

  – Representing a 2D signal as sum of 2D complex exponential signals

Illustration of Spatial Frequency



$f(x,y)=\sin(10\pi x)$

$f_x=5, f_y=0, f_m=5, \phi=0$

f_x : unit= 1 cycle/image-width

f_y : unit=1 cycle/image-height

$f(x,y)=\sin(10\pi x-20\pi y)$

$f_x=5, f_y=-10, f_m=\sqrt{125}, \phi=\text{atan}(-2)$

Example 1

$$f(x,y) = \sin 4\pi x + \cos 6\pi y$$

$$F\{\sin 4\pi x\} = \iint \sin 4\pi x e^{-j2\pi(ux+vy)} dx dy$$

$$= \int \sin 4\pi x e^{-j2\pi ux} dx \int e^{-j2\pi vy} dy$$

$$= \int \sin 4\pi x e^{-j2\pi ux} dx\, \delta(v)$$

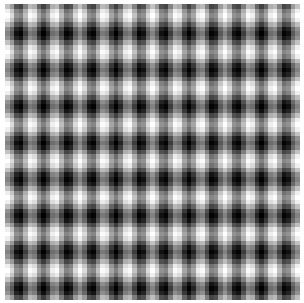
$$= \frac{1}{2j}(\delta(u-2) - \delta(u+2))\delta(v)$$

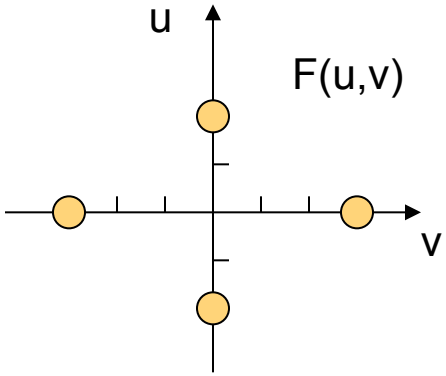
$$= \frac{1}{2j}(\delta(u-2,v) - \delta(u+2,v))$$

$$where \quad \delta(x,y) = \delta(x)\delta(y) = \begin{cases} \infty, & x = y = 0 \\ 0, & otherwise \end{cases}$$

$$Likewise, \quad F\{\cos 6\pi y\} = \frac{1}{2}(\delta(u,v-3) + \delta(u,v+3))$$

f(x,y)





Example 2

$$f(x,y)=\sin(2\pi x+3\pi y)=\frac{1}{2j}\left(e^{j(2\pi x+3\pi y)}-e^{-j(2\pi x+3\pi y)}\right)$$

$$F\left\{e^{j(2\pi x+3\pi y)}\right\}=\iint e^{j(2\pi x+3\pi y)}e^{-j2\pi(xu+yv)}dxdy$$

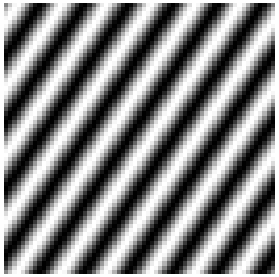
$$=\int e^{j2\pi x}e^{-j2\pi ux}dx\int e^{j3\pi y}e^{-j2\pi yv}dy$$

$$=\delta(u-1)\delta(v-\frac{3}{2})=\delta(u-1,v-\frac{3}{2})$$

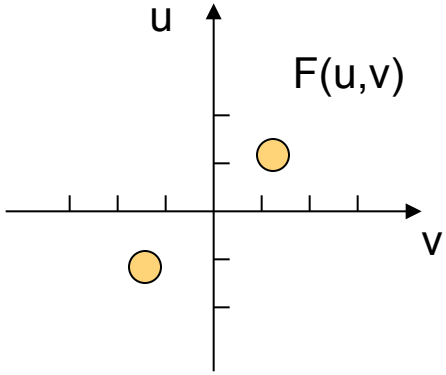
$$Likewise,\quad F\left\{e^{-j(2\pi x+3\pi y)}\right\}=\delta(u+1,v+\frac{3}{2})$$

$$Therefore,$$

$$F\{\sin(2\pi x+3\pi y)\}=\frac{1}{2j}\left(\delta(u-1,v-\frac{3}{2})-\delta(u+1,v+\frac{3}{2})\right)$$



```
[X,Y]=meshgrid(-2:1/16:2,-2:1/16:2);
f=sin(2*pi*X+3*pi*Y);
imagesc(f); colormap(gray)
Truesize, axis off;
```



Properties of 2D FT (1)

- Linearity

$$F\{a_1 f_1(x,y)+a_2 f_2(x,y)\}=a_1 F\{f_1(x,y)\}+a_2 F\{f_2(x,y)\}$$

- Translation

$$f(x-x_0,y-y_0) \quad \Leftrightarrow \quad F(u,v)e^{-j2\pi(x_0 u+y_0 v)},$$

$$f(x,y)e^{j2\pi(u_0 x+v_0 y)} \quad \Leftrightarrow \quad F(u-u_0,v-v_0)$$

- Conjugation

$$f^*(x,y) \quad \Leftrightarrow \quad F^*(-u,-v)$$

Properties of 2D FT (2)

- Symmetry

$$f(x,y)\ is\ real \quad \Leftrightarrow \quad |F(u,v)| = |F(-u,-v)|$$

- Convolution
  - Definition of convolution

$$f(x,y) \otimes g(x,y) = \iint f(x-\alpha, y-\beta) g(\alpha,\beta) d\alpha d\beta$$

  - Convolution theory

$$f(x,y) \otimes g(x,y) \quad \Leftrightarrow \quad F(u,v)G(u,v)$$

We will describe 2D convolution later!

Separability of 2D FT and Separable Signal

- Separability of 2D FT

$$F_2\{f(x,y)\} = F_y\{F_x\{f(x,y)\}\} = F_x\{F_y\{f(x,y)\}\}$$

  - where F_x, F_y are 1D FT along x and y.
  - one can do 1DFT for each row of original image, then 1D FT along each column of resulting image
- Separable Signal
  - $f(x,y) = f_x(x)f_y(y)$
  - $F(u,v) = F_x(u)F_y(v)$,
    - where $F_x(u) = F_x\{f_x(x)\}$, $F_y(u) = F_y\{f_y(y)\}$
  - For separable signal, one can simply compute two 1D transforms and take their product!

Example 1

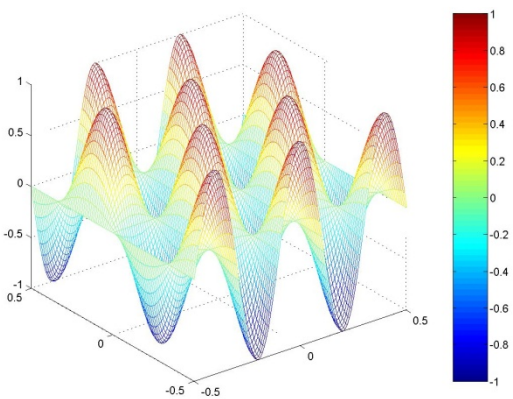
$$f(x,y) = \sin(3\pi x)\cos(5\pi y)$$

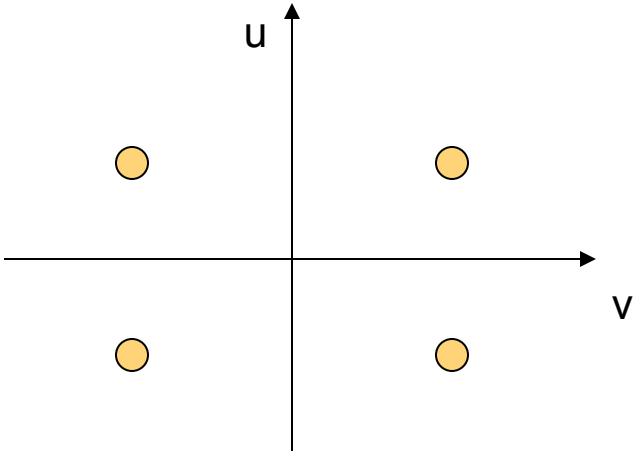
$$f_x(x) = \sin(3\pi x) \quad \Leftrightarrow \quad F_x(u) = \frac{1}{2j}\left(\delta(u-3/2) - \delta(u+3/2)\right)$$

$$f_y(y) = \cos(5\pi y) \quad \Leftrightarrow \quad F_y(v) = \frac{1}{2}\left(\delta(v-5/2) + \delta(v+5/2)\right)$$

$$F(u,v) = Fx(u)Fy(v)$$

$$= \frac{1}{4j}\left(\delta(u-\frac{3}{2}, v-\frac{5}{2}) - \delta(u+\frac{3}{2}, v-\frac{5}{2}) + \delta(u-\frac{3}{2}, v+\frac{5}{2}) - \delta(u+\frac{3}{2}, v+\frac{5}{2})\right)$$

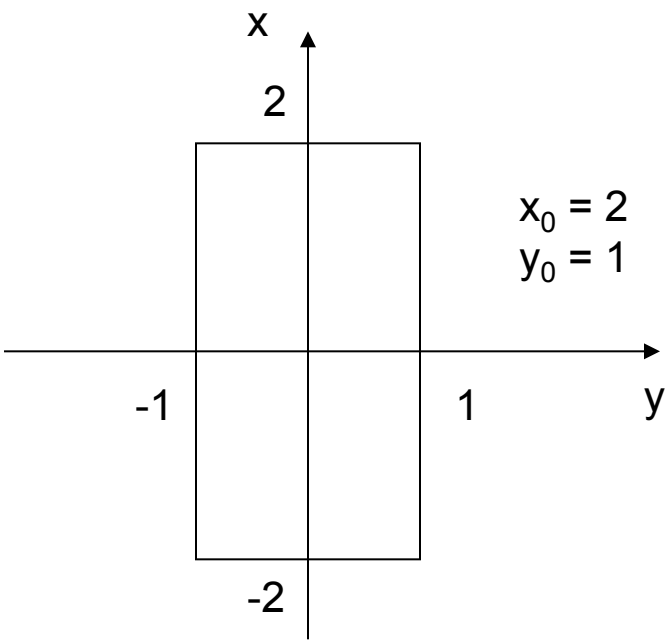


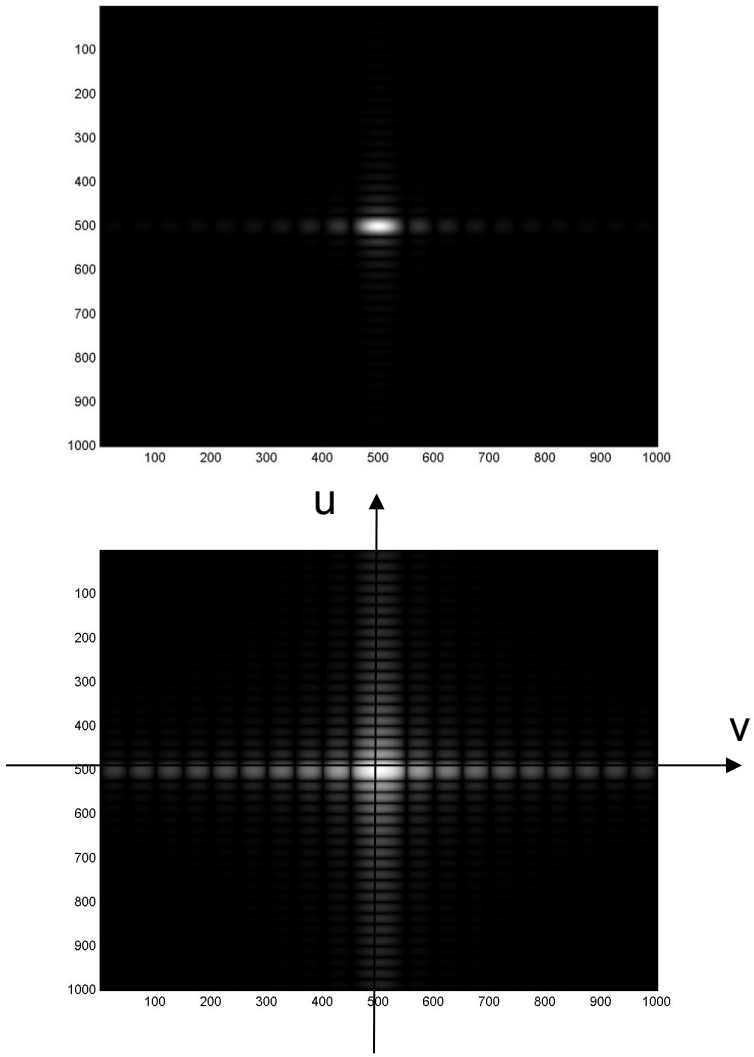


Example 2

$$f(x,y)=\begin{cases}1, & |x|\le x_0, |y|\le y_0\\ 0, & otherwise\end{cases} \Rightarrow$$

$$F(u,v)=4x_0y_0\,\mathrm{sinc}(2x_0u)\,\mathrm{sinc}(2y_0v)$$





w/ logrithmic mapping

Example 3: Gaussian Signal

- Still a Gaussian Function!
- 1D Gaussian Signal

$$\exp\left\{-\frac{x^2}{2\sigma^2}\right\} \Leftrightarrow \exp\left\{-\frac{u^2}{2\beta^2}\right\}, \beta = \frac{1}{2\pi\sigma}$$

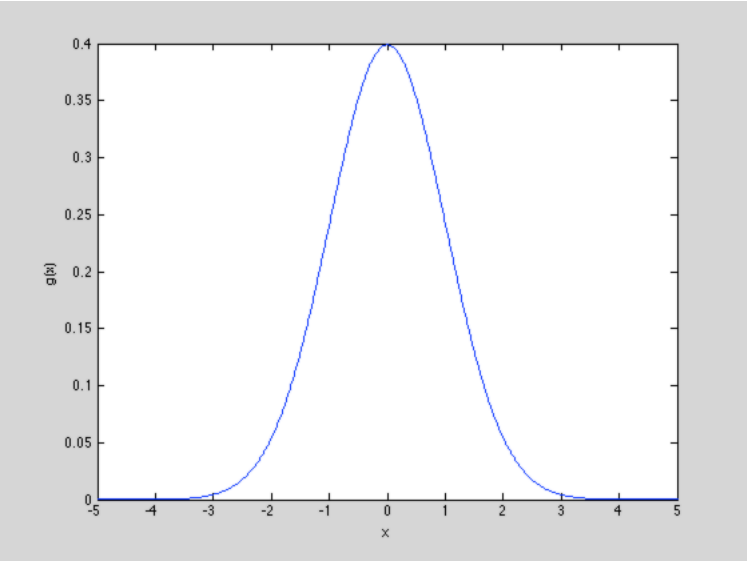
- 2D Gaussian Signal

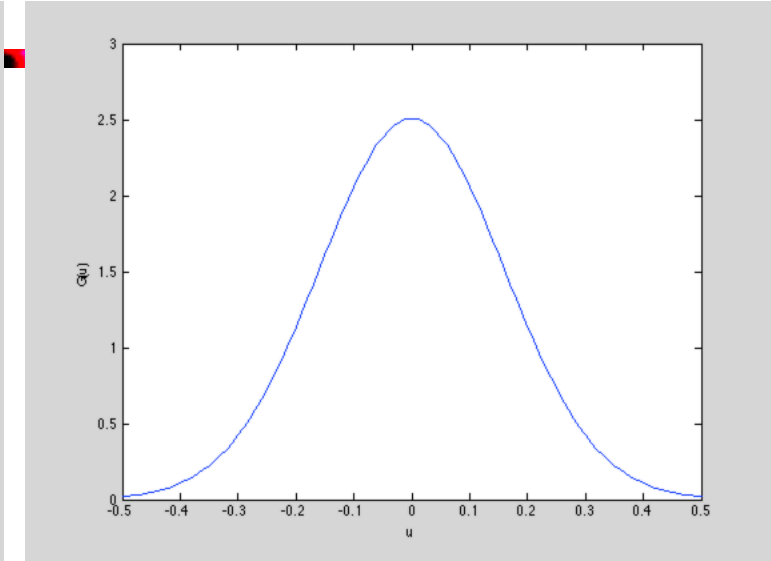
$$\exp\left\{-\frac{x^2+y^2}{2\sigma^2}\right\} = \exp\left\{-\frac{x^2}{2\sigma^2}\right\}\exp\left\{-\frac{y^2}{2\sigma^2}\right\}$$
$$\leftrightarrow \exp\left\{-\frac{u^2}{2\beta^2}\right\}\exp\left\{-\frac{v^2}{2\beta^2}\right\} = \exp\left\{-\frac{u^2+v^2}{2\beta^2}\right\}, \beta = \frac{1}{2\pi\sigma}$$

- Note that STD β in freq. inversely related to STD σ in space

Illustration of Gaussian Signal

σ=1



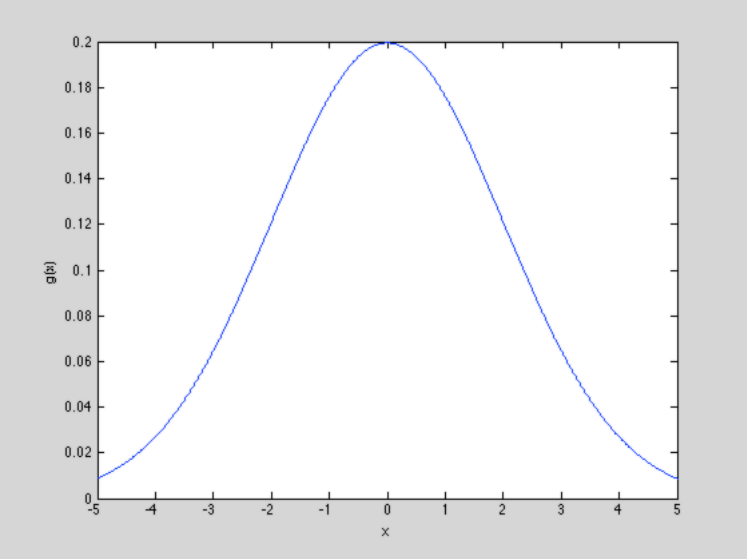


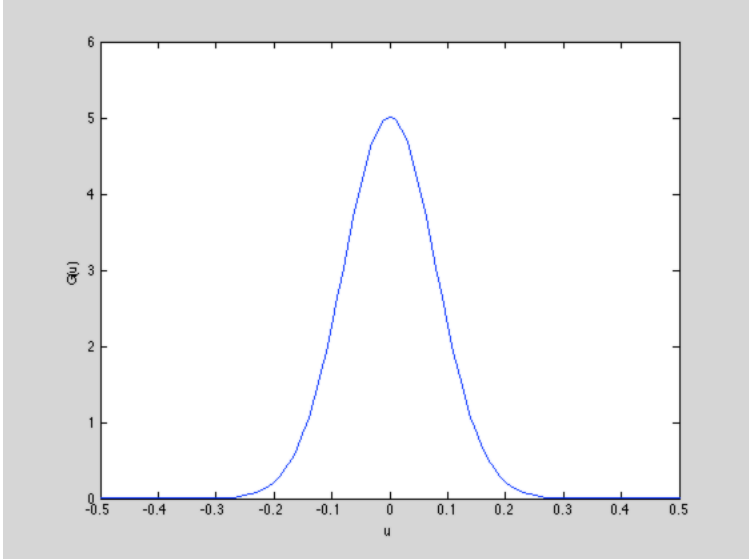
β=0.16

g(x)

G(u)

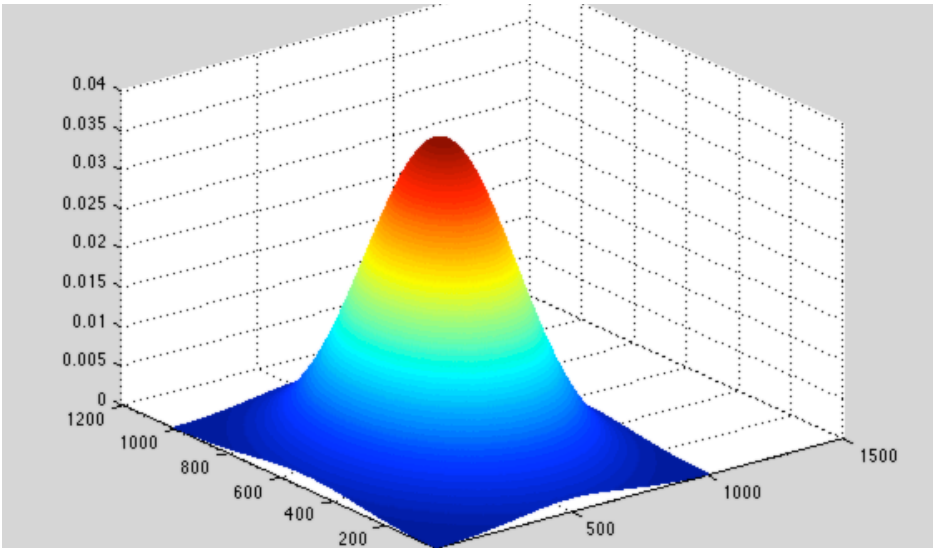
σ=2





β=0.08

2D Guaussian Function



Surface plot (surf( ))

Plot as image

Important Transform Pairs

- All following signals are separable and can be proved by applying the separability of the CSFT

$$f(x,y)=1 \quad \Leftrightarrow \quad F(u,v)=\delta(u,v), \quad where\,\delta(u,v)=\delta(u)\delta(v)$$

$$f(x)=e^{j2\pi(f_1x+f_2y)} \quad \Leftrightarrow \quad F(u)=\delta(u-f_1,v-f_2)$$

$$f(x,y)=\cos(2\pi(f_1x+f_2y)) \quad \Leftrightarrow \quad F(u)=\frac{1}{2}\left(\delta(u-f_1,v-f_2)+\delta(u+f_1,v+f_2)\right)$$

$$f(x,y)=\sin(2\pi(f_1x+f_2y)) \quad \Leftrightarrow \quad F(u)=\frac{1}{2j}\left(\delta(u-f_1,v-f_2)-\delta(u+f_1,v+f_2)\right)$$

$$f(x,y)=\begin{cases} 1, & |x|<x_0, |y|<y_0 \\ 0, & otherwise \end{cases} \quad \Leftrightarrow$$

$$F(u,v)=\frac{\sin(2\pi x_0 u)}{\pi u}\frac{\sin(2\pi y_0 v)}{\pi v}=4x_0y_0\,\mathrm{sinc}(2x_0u)\mathrm{sinc}(2y_0v)$$

$$where\ \mathrm{sinc}(t)=\frac{\sin(\pi t)}{\pi t}$$

$$\exp\left\{-\frac{x^2+y^2}{2\sigma^2}\right\} \Leftrightarrow \exp\left\{-\frac{u^2+v^2}{2\beta^2}\right\}, \beta=\frac{1}{2\pi\sigma}$$

- Constant ⇔ impulse at (0,0) freq.
- Complex exponential ⇔ impulse at a particular 2D freq.
- 2D box ⇔ 2D sinc function
- Gaussian ⇔ Gaussian

Rotation

- Let $x = r\cos\theta, \quad y = r\sin\theta, \quad u = \rho\cos\omega, \quad v = \rho\sin\omega.$
- 2D FT in polar coordinate (r, θ) and (ρ, Ø)

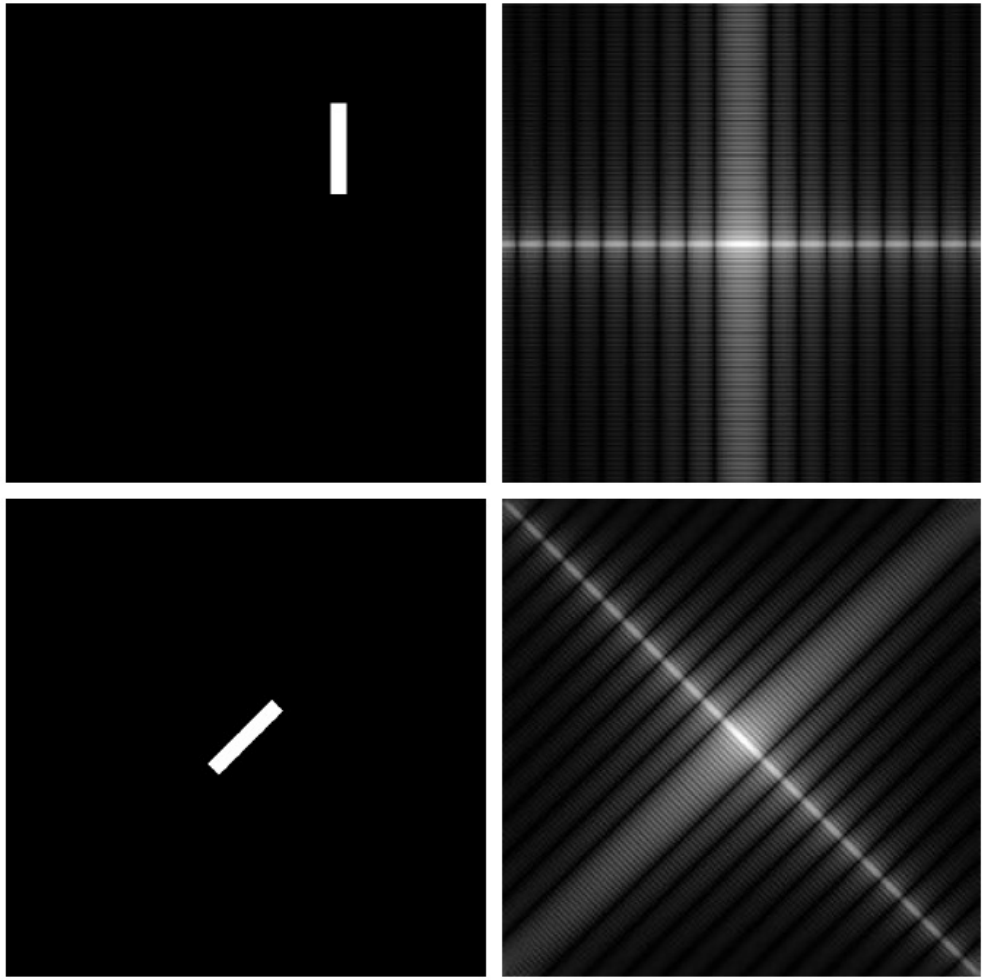
$$F(\rho,\phi) = \int_0^{\infty}\int_0^{2\pi} f(r,\theta)e^{-j2\pi(r\cos\theta\,\rho\cos\phi + r\sin\theta\,\rho\sin\phi)} r\,dr\,d\theta$$
$$= \iint f(r,\theta)e^{-j2\pi r\rho\cos(\theta-\phi)} r\,dr\,d\theta$$

- Property

$$f(r,\theta+\theta_0) \quad \Leftrightarrow \quad F(\rho,\phi+\theta_0)$$

- Proof: Homework!

Example of Rotation



a b
c d

FIGURE 4.25 (a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

1D Fourier Transform For Discrete Time Sequence (DTFT) (Review)

- f(n) is a 1D discrete time sequence
- Forward Transform

$$F(u) = \sum_{n=-\infty}^{\infty} f(n) e^{-j2\pi un}$$

- Inverse Transform

$$f(n) = \int_{-1/2}^{1/2} F(u) e^{j2\pi un} du$$

- Representing f(n) as weighted sum of many complex sinusoidal signals with frequency u, F(u) is the weight
- F(u) indicate the “amount” of sinusoidal component with freq. u in signal f(n)
- |u| = digital frequency = the number of cycles per integer sample. Period = 1/|u| (must be equal or greater than 2 samples-> |u|<=1/2)

Properties unique for DTFT

- Periodicity
  - F(u) = F(u+1)
  - The FT of a discrete time sequence is only considered for u ϵ (-½ , ½), and u = ±½ is the highest discrete frequency
- Symmetry for real sequences

$$f(n) = f^*(n) \quad \Leftrightarrow \quad F(u) = F^*(-u)$$

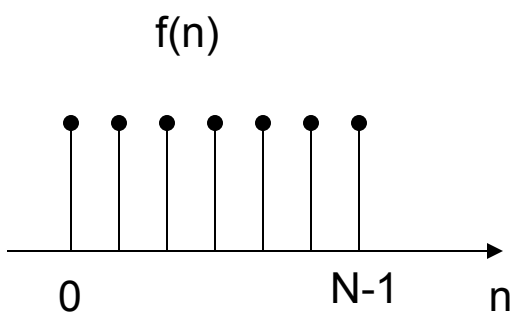
$$\Rightarrow \quad |F(u)| = |F(-u)|$$

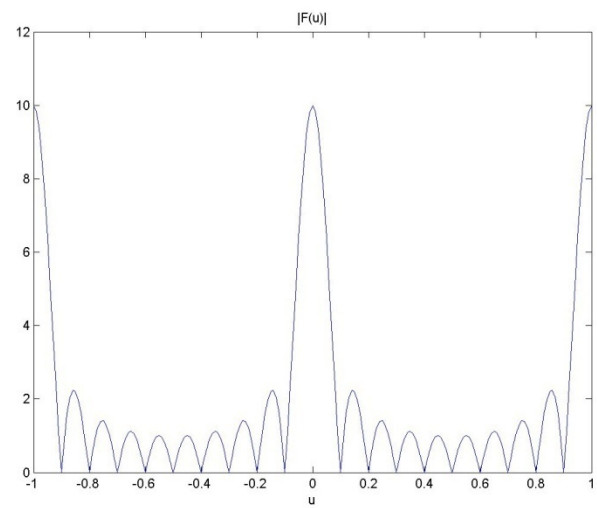
$$\Rightarrow \quad |F(u)| \; is \; symmetric$$

Example

$$f(n) = \begin{cases} 1, & n = 0,1,\ldots,N-1; \\ 0, & others \end{cases}$$

$$F(u) = \sum_{n=0}^{N-1} e^{-j2\pi nu} = \frac{1-e^{-j2\pi uN}}{1-e^{-j2\pi u}} = e^{-j\pi(N-1)u}\,\frac{\sin 2\pi u(N/2)}{\sin 2\pi u(1/2)}$$





N=10

There are N/2 zeros in (0, ½], 1/N apart

1D Discrete Fourier Transform (DFT)

- DSFT of N-pt sequence

$$F(u)=\sum_{n=0}^{N-1} f(n)e^{-j2\pi un}$$

- N-pt DFT of N-pt sequence

$$F_N(k)=\sum_{n=0}^{N-1} f(n)e^{-j2\pi\frac{kn}{N}}=F\left(u=\frac{k}{N}\right)$$

- DFT is the sampled version of DTFT with samples at 0, 1/N, …, (N-1)/N.
- FFT: Fast algorithm for computing DFT
  - Direct computation takes N^2 operations
  - FFT takes ~N log(N) operations!

Discrete Space Fourier Transform (DSFT) for Two Dimensional Signals

- Let f(m,n) represent a 2D sequence
- Forward Transform

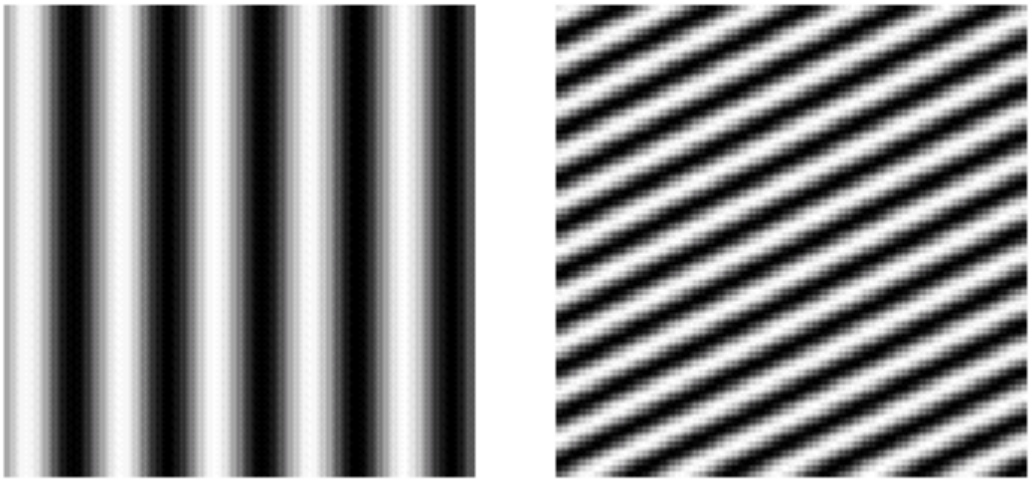
$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m,n) e^{-j2\pi(mu+nv)}$$

- Inverse Transform

$$f(m,n) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} F(u,v) e^{j2\pi(mu+nv)} du dv$$

- Representing an image as weighted sum of many 2D complex sinusoidal images
- u: number of cycles per vertical sample (vertical freq.)
- v: number of cycles per horizontal sample (horizontal freq)

Spatial Frequency for Digital Images



(a) (b)

Figure 2.1 Two-dimensional sinusoidal signals: (a) $(f_x, f_y) = (5, 0)$; (b) $(f_x, f_y) = (5, 10)$. The horizontal and vertical units are the width and height of the image, respectively. Therefore, $f_x = 5$ means that there are five cycles along each row.

$$f_s = \sqrt{125}, \quad \varphi = \arctan(2)$$

If the image has 256x256 pixels, fx=5 cycles per width (analog frequency) -> u= 5 cycles/256 pixels =5/256 cycles/sample (digital frequency)

When both horizontal and vertical frequency are non-zero, we see directional patterns. fs is the frequency along the direction with the maximum change (orthogonal to the lines)

Note that 1/u may not correspond to integer.

If u=a/b, digital period= b. Eg u=3/8, Digital Period = 8

Periodicity

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m,n) e^{-j2\pi(mu+nv)}$$

- *F(u,v)* is periodic in u, v with period 1, i.e., for all integers k, l:
  - F(u+k, v+l) = F(u, v)
- To see this consider

$$e^{-j2\pi(m(u+k)+n(v+l))}$$

$$= e^{-j2\pi(mu+nv)} e^{-j2\pi(mk+nl)}$$

$$= e^{-j2\pi(mu+nv)}$$

Example: Delta Function

- Fourier transform of a delta function

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(m,n) e^{-j2\pi(mu+nv)} = 1$$

$$\delta(m,n) \Leftrightarrow 1$$

  – Delta function contains all frequency components with equal weights!

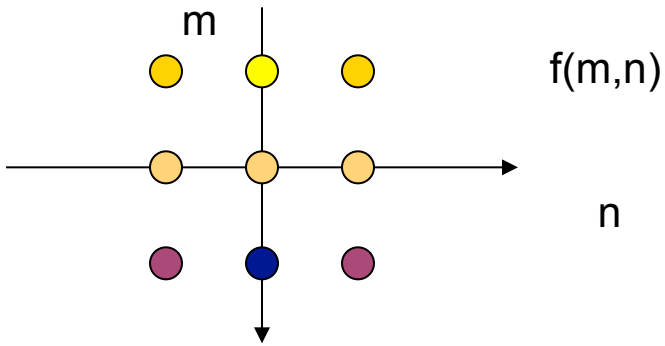
- Inverse Fourier transform of a delta function

$$f(m,n) = \int_{-0.5}^{0.5} \int_{-0.5}^{0.5} \delta(u,v) e^{j2\pi(mu+nv)} du dv = 1$$

$$1 \Leftrightarrow \delta(u,v)$$

Example

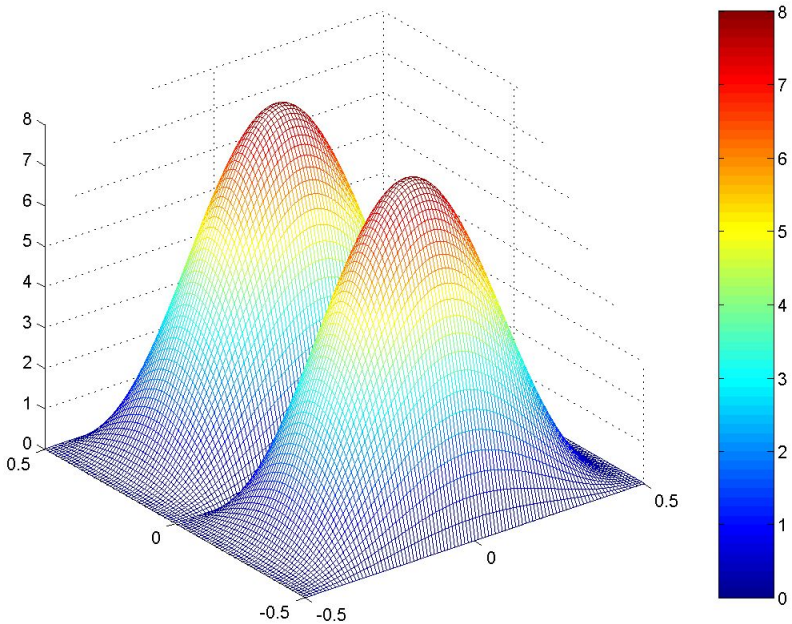
$$f(m,n)=\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

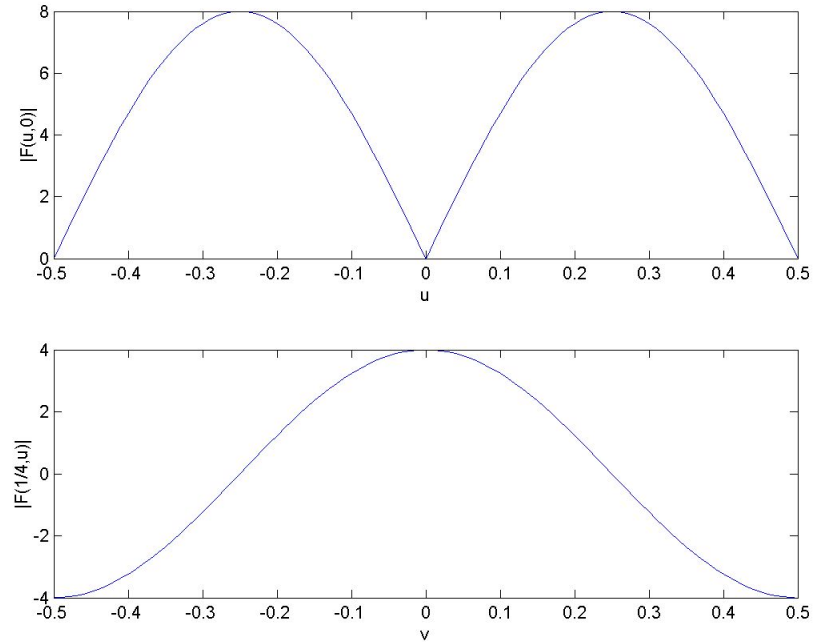


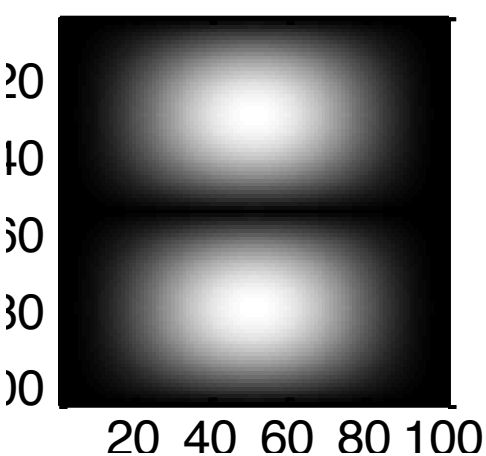
$$\begin{aligned} F(u,v) &= 1e^{-j2\pi(-1*u-1*v)} + 2e^{-j2\pi(-1*u+0*v)} + 1e^{-j2\pi(-1*u+1*v)} \\ &\quad -1e^{-j2\pi(1*u-1*v)} - 2e^{-j2\pi(1*u+0*v)} - 1e^{-j2\pi(1*u+1*v)} \\ &= j2\sin 2\pi u e^{j2\pi v} + j4\sin 2\pi u + j2\sin 2\pi u e^{-j2\pi v} \\ &= j4\sin 2\pi u(\cos 2\pi v + 1) \end{aligned}$$

Note: This signal is low pass in the horizontal direction v (weighted average), and band pass in the vertical direction u (difference).

Graph of F(u,v)








```
du = [-0.5:0.01:0.5];
dv = [-0.5:0.01:0.5];
Fu = abs(sin(2 * pi * du));
Fv = cos(2 * pi * dv);
F = 4 * Fu' * (Fv + 1);
mesh(du, dv, F);
colorbar;
Imagesc(F);
colormap(gray); truesize;
```

Using MATLAB freqz2:

```
f=[1,2,1;0,0,0;-1,-2,-1];
freqz2(f)
```

DSFT vs. 2D DFT

- 2D DSFT

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m,n) e^{-j2\pi(mu+nv)}$$

- 2D DFT

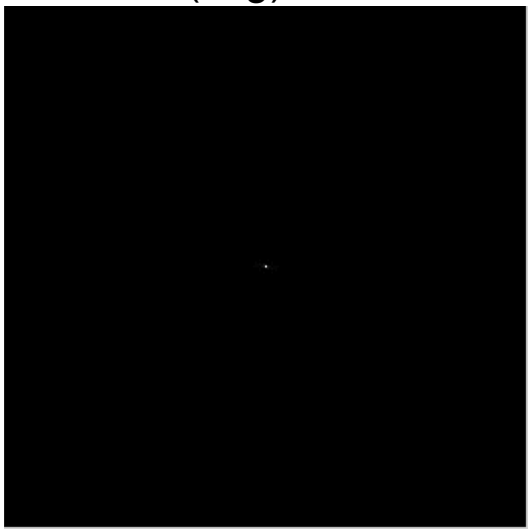
$$F(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) e^{-j2\pi(mk/M+nl/N)}$$

- (MxN) point 2D DFT are samples of DSFT for images of size MxN at u=m/M, v=n/N. Can be computed using FFT algorithms. 1D FFT along rows, then 1D FFT along columns

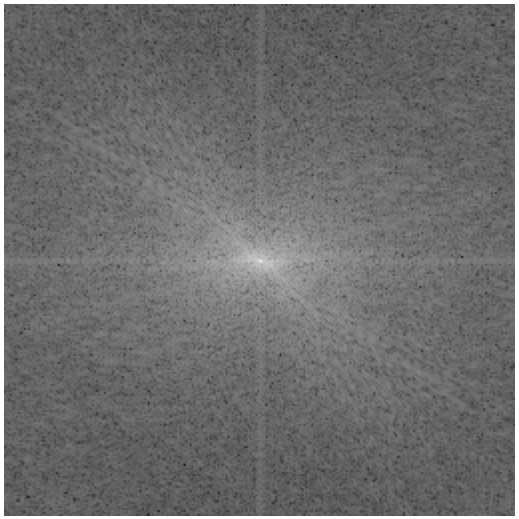
Display of DFT of Images

Imshow(img)

MFimg=abs(fft2(img)), imshow(MFimg,[ ])



SMFimg=fftshift(MFimg)
imshow(SMFimg,[ ])

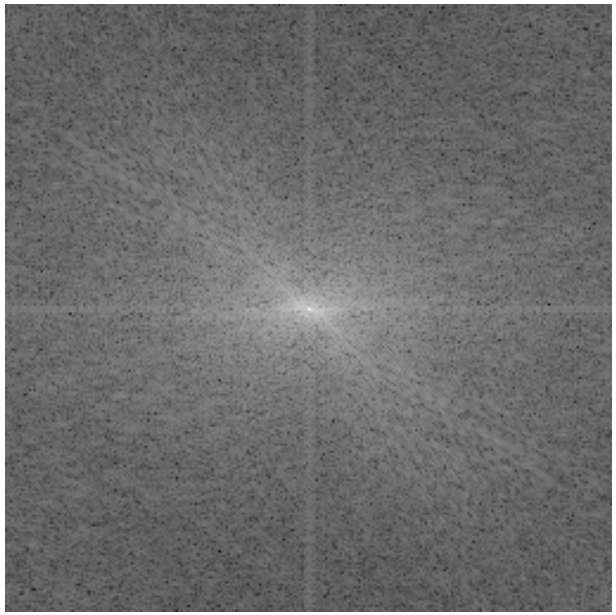


LSMFimg=log(SMFimg+1);
Imshow(LSMFimg,[ ])

fftshift to shift the (0,0) freq. to center

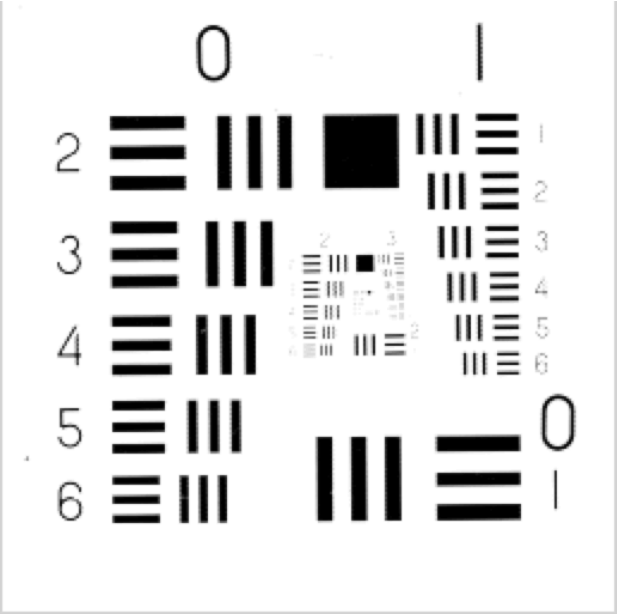
Log mapping to enhance contrast

DFT of Typical Images

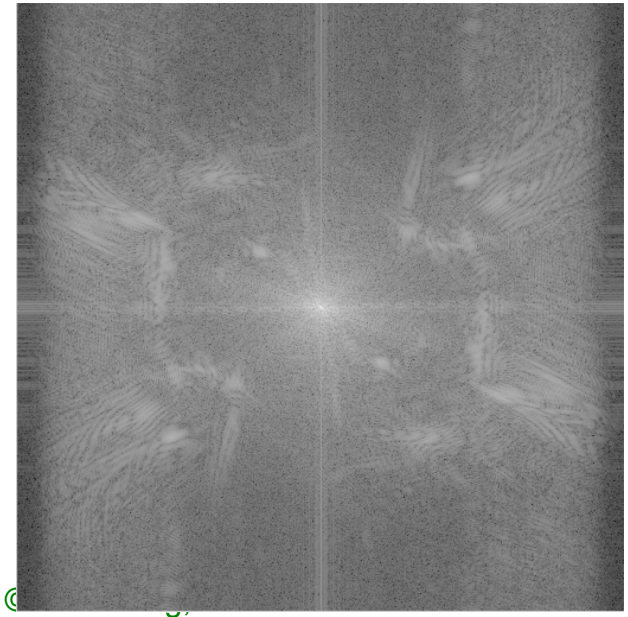


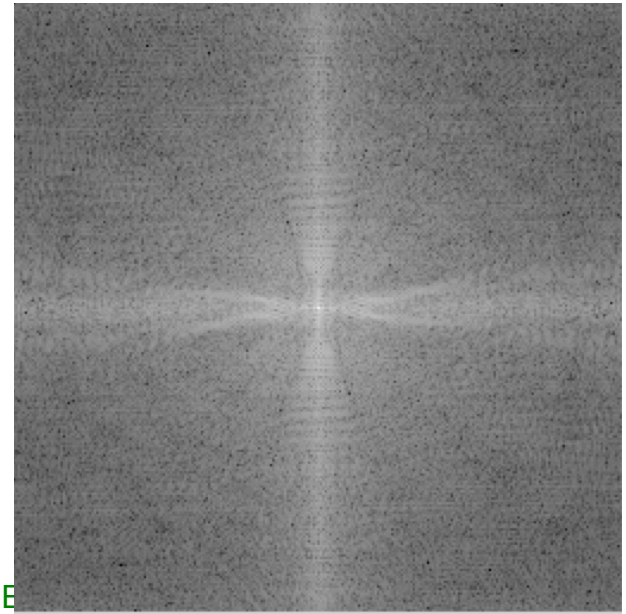
How to interpret the DSFT image?
Why is it bright at center and has some line structures?

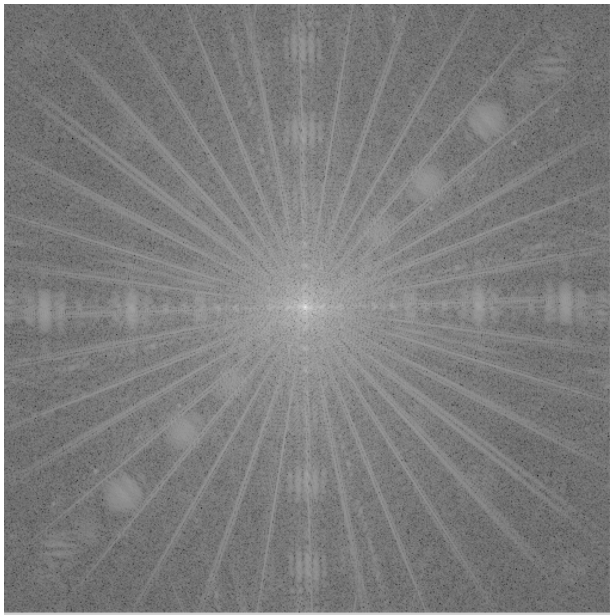
|F(u,v)| (obtained using 2D FFT)
ff=abs(fft2(f));
imagesc(fftshift(log(ff+1)));
Log mapping to enhance contrast
Fftshift to shift the (0,0) freq. to center

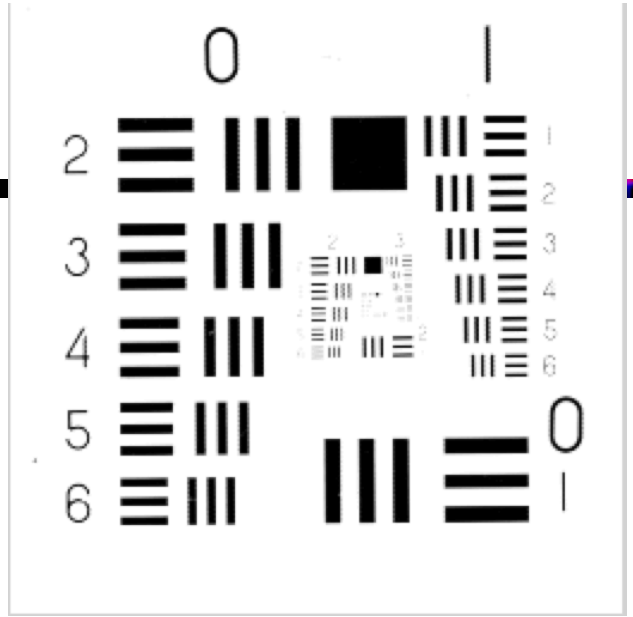


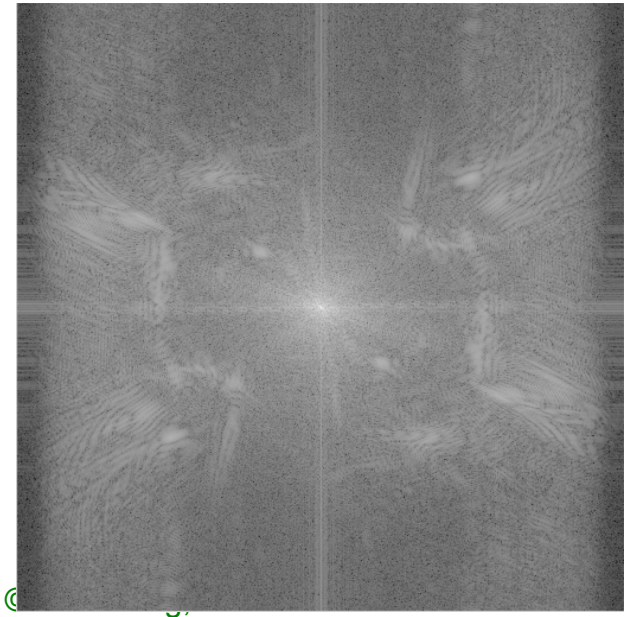
Which one below is the DFT of which one above?

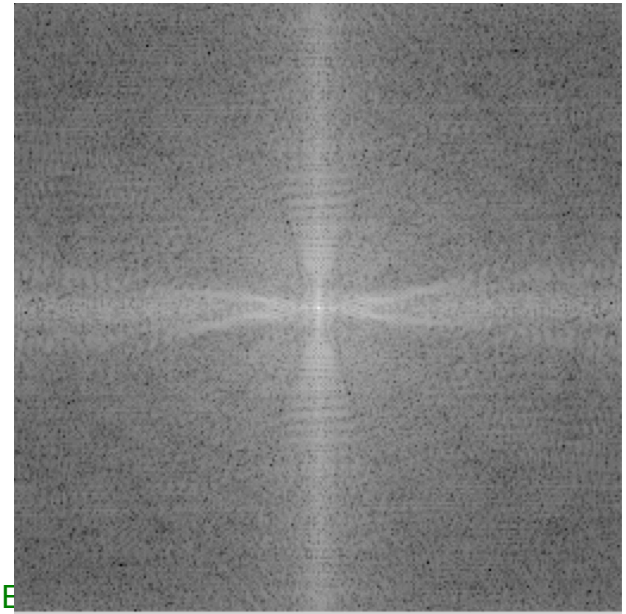


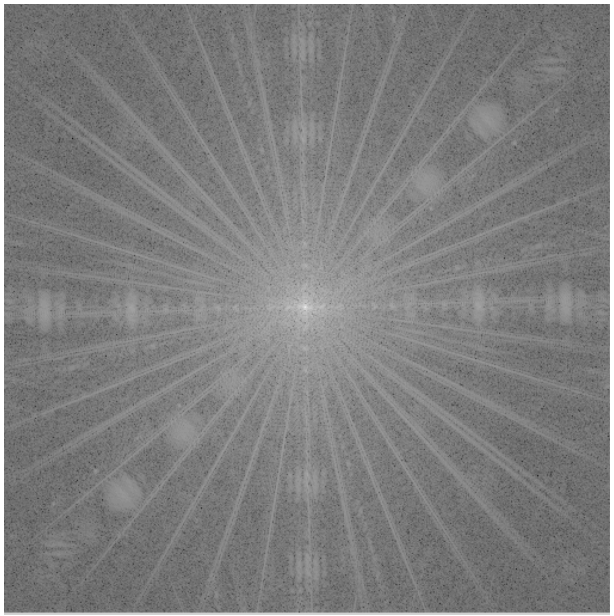





0
1
2
3
4
5
6
1
2
3
4
5
6
0
1







DSFT of Separable Signals

- Separable signal:
  - $h(m,n)=hx(m)\ hy(n)$
- 2D DSFT of separable signal = product of 1D DSFT of each 1D component
  - $H(u,v)= Hx(u)\ Hy(v)$
  - $Hx(u)$: 1D FT of hx
  - $Hy(v)$: 1D FT of hy

DSFT of Special Signals

- Constant <-> pulse at (0,0)
- Rectangle <-> 2D digital sinc
- Sinc <-> Rectangle (ideal low pass)
- Complex exponential with freq (u0,v0) <-> pulse at (u0,v0)
- …
- Can be shown easily by making use of the fact that the signal is separable

Using Separable Processing to Compute DSFT

- 3x3 averaging filter

$$H = 1/9\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 1/3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}1/3 = h_1 h_1^T$$

- Recognizing that the filter is separable

$$1/3\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} <-> H_1(v) = (1e^{j2\pi v} + 1 + 1e^{-j2\pi v})/3 = (1 + 2\cos 2\pi v)/3$$

$$H(u,v) = H_1(u)H_1(v) = (1+\cos 2\pi u)(1+\cos 2\pi v)/9$$

Using Separable Processing to Compute DSFT

- Another example

$$H = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = h_x h_y^T$$

- Recognizing that the filter is separable

$$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} <-> Fy(v) = 1e^{j2\pi v} + 0 + (-1)e^{-j2\pi v} = 2j\sin 2\pi v$$

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} <-> Fx(u) = 1e^{j2\pi u} + 2 + e^{-j2\pi u} = 2 + 2\cos 2\pi u$$

$$F(u,v) = Fx(u)Fy(v) = 4j(1+\cos 2\pi u)\sin 2\pi v$$

What about rotation?

- No theoretical proof, however, roughly it is still true
- Rotation in space <-> Rotation in freq.
- Example:
- $f(m,n)=\delta(m)$ (horizontal line) <-> $F(u,v)=\delta(v)$ (vertical line)
- What about $f(m,n)=\delta(m-n)$ (diagonal line)?
  - <-> $F(u,v)=\delta(u+v)$ (antidiagonal line!)
  - Homework

Linear Convolution over Continuous Space

- 1D convolution

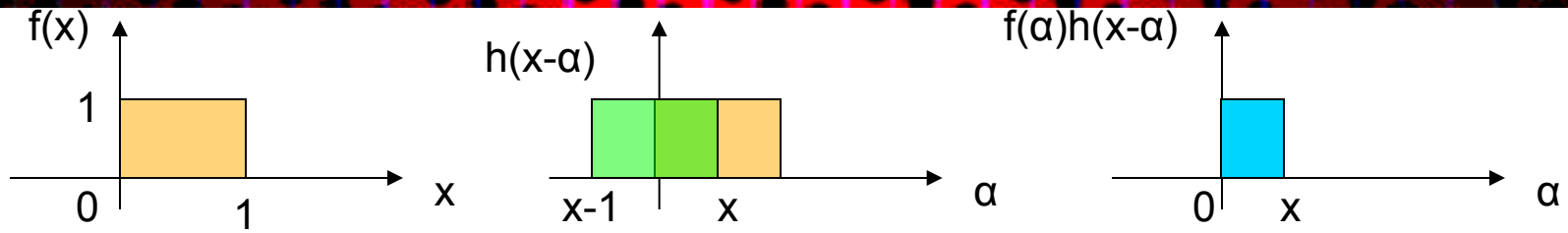
$$f(x)*h(x)=\int_{-\infty}^{\infty} f(x-\alpha)h(\alpha)d\alpha=\int_{-\infty}^{\infty} f(\alpha)h(x-\alpha)d\alpha$$

$$f(x)*\delta(x)=f(x),\quad f(x)*\delta(x-x_0)=f(x-x_0)$$

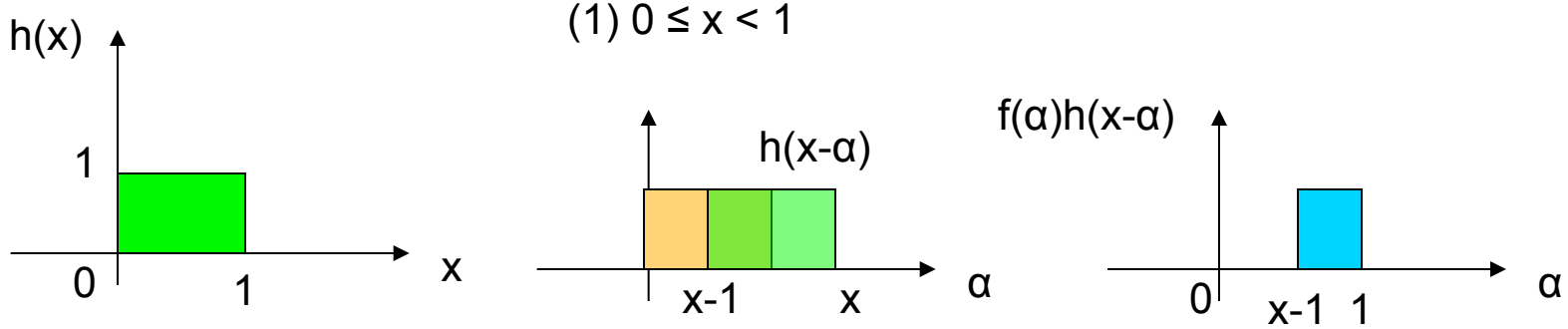
- 2D convolution

$$f(x,y)*h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\alpha,y-\beta)h(\alpha,\beta)d\alpha d\beta$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)h(x-\alpha,y-\beta)d\alpha d\beta$$

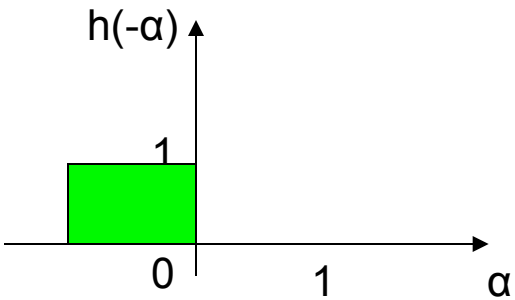
Examples of 1D Convolution

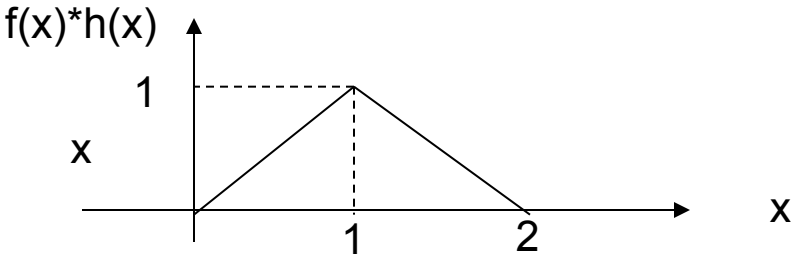


(1) $0 \le x < 1$

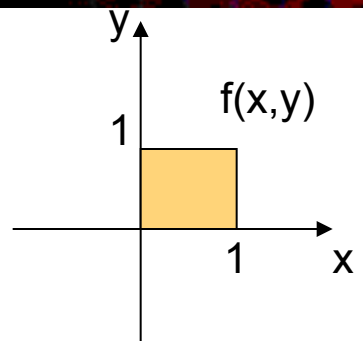


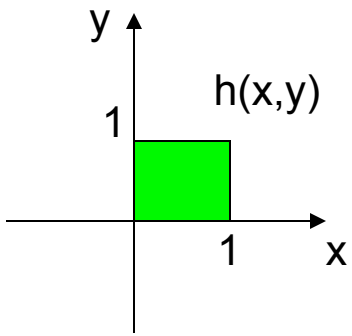
(2) $1 \le x < 2$

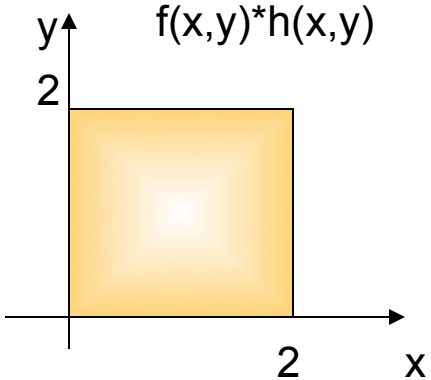


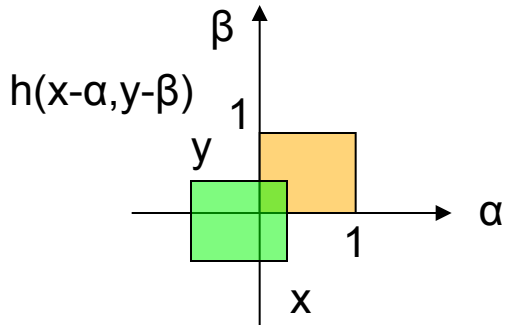


Example of 2D Convolution

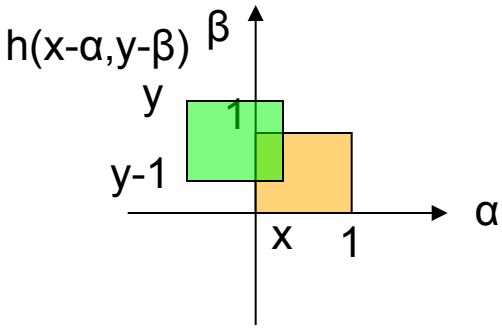




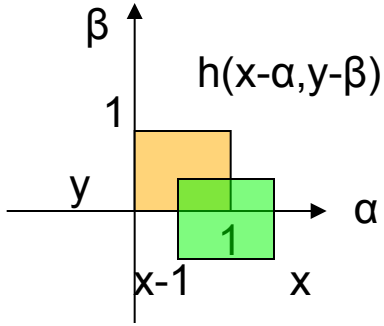




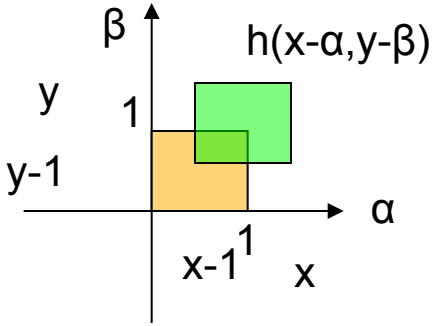
(1) 0<x≤1, 0<y≤1
g(x,y)=x*y



(2) 0<x≤1, 1<y≤2
g(x,y)=x*(2-y)



(3) 1<x≤2, 0<y≤1
g(x,y)=(2-x)*y



(4) 1<x≤2, 1<y≤2
g(x,y)=(2-x)*(2-y)

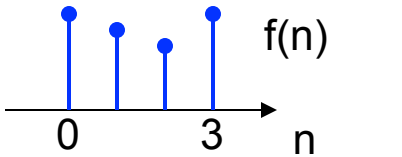
Convolution of 1D Discrete Signals

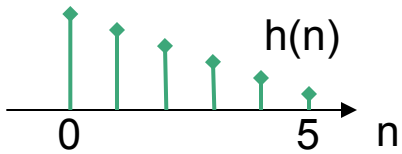
- Definition of convolution

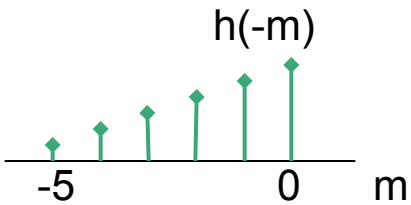
$$f(n) * h(n) = \sum_{m=-\infty}^{\infty} f(n-m)h(m) = \sum_{m=-\infty}^{\infty} f(m)h(n-m)$$

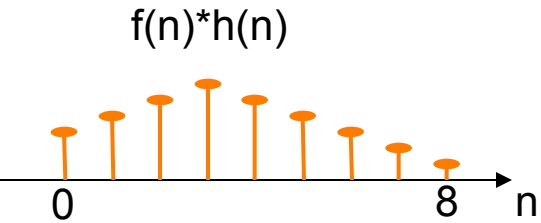
- The convolution with h(n) can be considered as the weighted average in the neighborhood of f(n), with the filter coefficients being the weights
  - sample f(n-m) is multiplied by h(m)
- The filter h(n) is the impulse response of the system (i.e. the output to an input that is an impulse)
- Signal length before and after filtering
  - Original signal length: N
  - Filter length: K
  - Filtered signal length: N+K-1

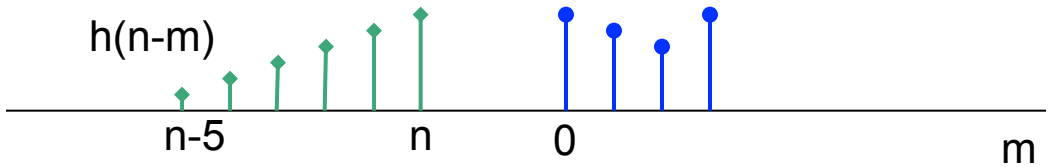
Example of 1D Discrete Convolution



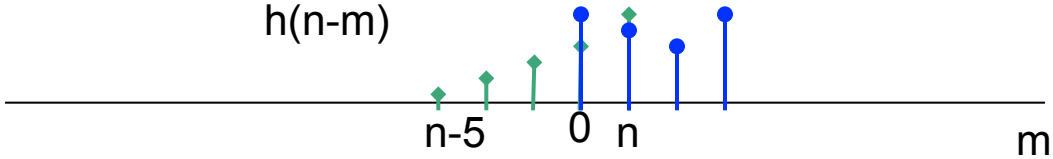




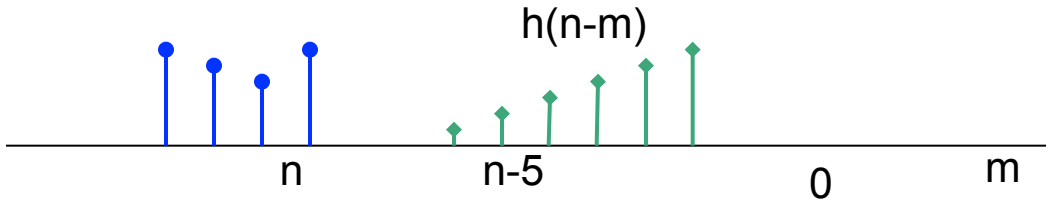




(a) $n < 0$, $g(n) = 0$



(b) $0 \le n \le 8$, $g(n) > 0$



(c) $n > 8$, $g(n) = 0$

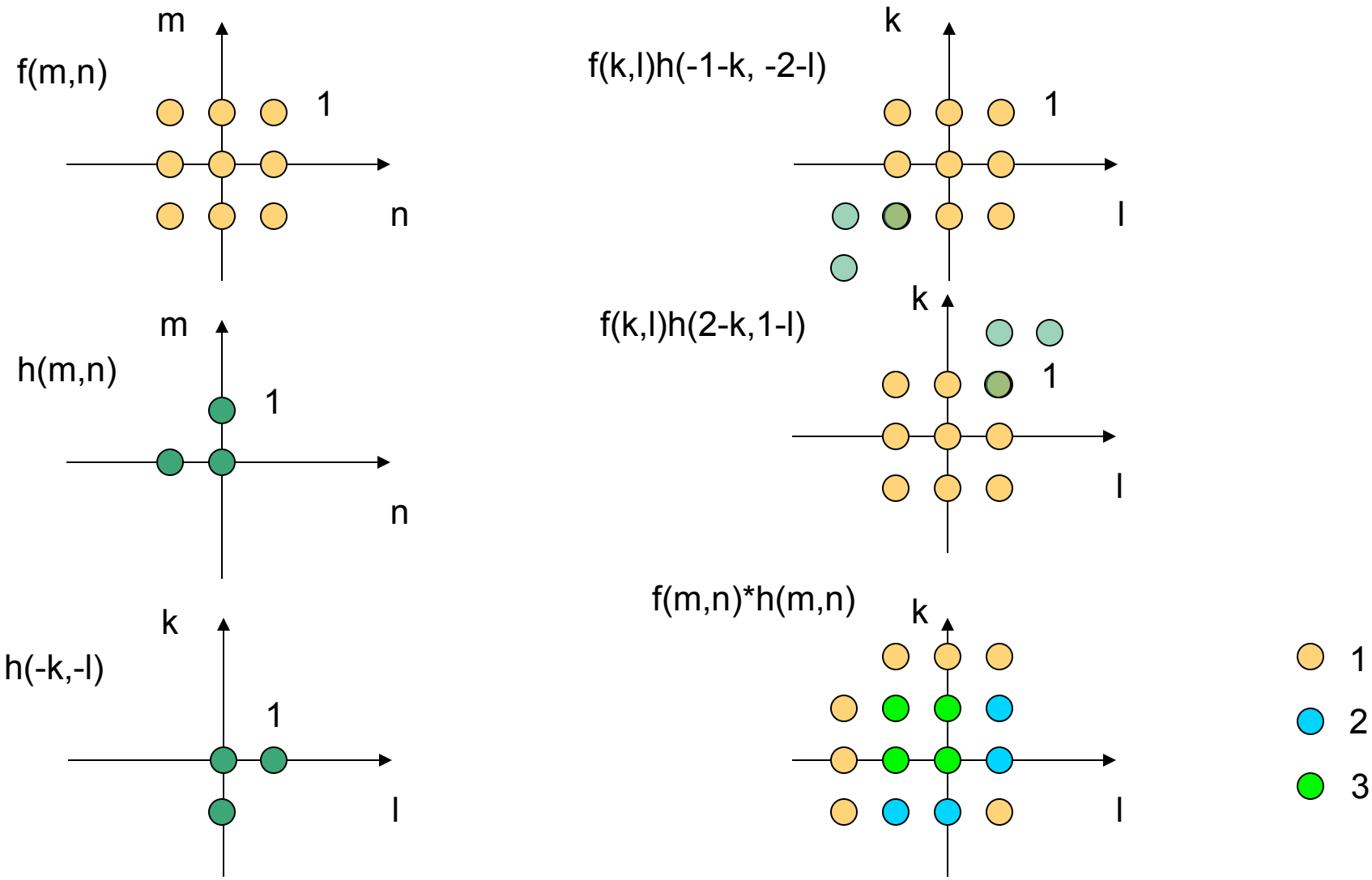
Convolution of 2D Discrete Signals

$$g(m,n) = f(m,n) * h(m,n) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f(m-k,n-l)h(k,l)$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f(k,l)h(m-k,n-l)$$

- Each new pixel g(m,n) is a weighted average of its neighboring pixels in the original image:
  - Pixel f(m-k,n-l) is weighted by h(k,l)
- We may use matrices to represent both signal (F) and filter (H) and use F*H to denote the convolution

Example of 2D Discrete Convolution



Example: Averaging and Weighted Averaging

100	100	100	100	100
100	200	205	203	100
100	195	200	200	100
100	200	205	195	100
100	100	100	100	100

$\frac{1}{9}\times$

1	1	1
1	1	1
1	1	1

$\frac{1}{16}\times$

1	2	1
2	4	2
1	2	1

100	100	100	100	100
100	144	167	145	100
100	167	200	168	100
100	144	166	144	100
100	100	100	100	100

100	100	100	100	100
100	156	176	158	100
100	174	201	175	100
100	156	175	156	100
100	100	100	100	100

Example

Original image

Average filtered image

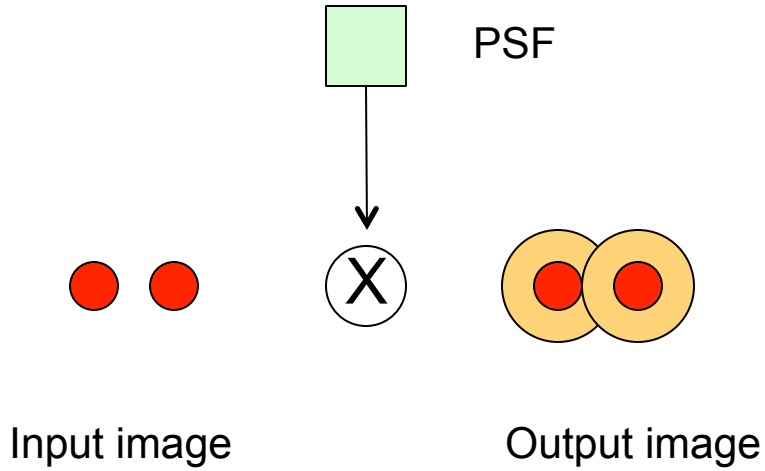
Weighted Average filtered image

What does h(m,n) mean?

- Any operation that is linear and shift invariant can be described by a convolution with a filter h(m,n)!
- h(m,n) is the impulse response of the system (i.e. output of the system to an impulse input)
- Better known as point spread function, indicating how a single point (i.e. an impulse) in the original image would be spread out in the output image

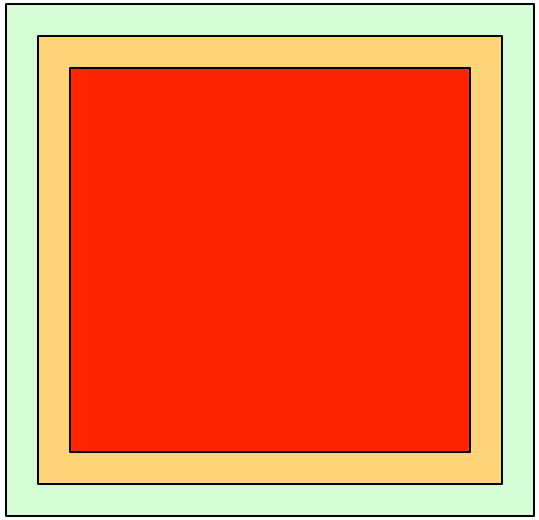
Point Spread Function

- The point spread function of an imaging system (e.g. a camera or a medical imaging system) describes the resolution of the system:
  - Two object points cannot be separated if they are closer than the support of the point spread function!



Boundary of Filtered Image

- An image of size MxN convolving with a filter of size KxL will yield an image of size (M+K-1,N+L-1)
- If the filter is symmetric with (2k +1)x(2k+1) samples, the convolved image should have an extra boundary of thickness k on each side outside the original image (outer boundary). The values along the outer boundary depend on the pixel values outside the original image
- Filtered values in the inner boundary of k pixels inside the original image also depend on the pixel values outside the original image



Orange+Red: original image size
Red: Valid part of the output image (does not depend on pixels outside the original image)
Orange: inner boundary
Green: outer boundary

Boundary Treatment: Zero Padding

- MxN image convolved with KxL filter -> (M+K-1)x(N+L-1) image
- Filtered values at the boundary of the output image depend on the assumed value of the pixels outside the original image
- Zero padding: Assuming pixel values are 0 outside the original image

0	0	0	0	0
0	200	205	203	0
0	195	200	200	0
0	200	205	195	0
0	0	0	0	0

$\frac{1}{9}\times$

1	1	1
1	1	1
1	1	1

Outer boundary

200/9	(200+205)/9	…	…	…
(200+195)/9	(200+205+195+200)/9	…	…	…
…	---	…	…	…
…	…	…	…	…
…	…	…	…	…

Actual image pixels

Extended pixels

Inner boundary

Boundary Treatment: Symmetric Extension

- Assuming pixels values outside the image are the same as their mirroring pixels inside the image
  - Lead to less discontinuity in the filtered image along the outer and inner boundaries

200	200	205	203	203
200	200	205	203	203
195	195	200	200	200
200	200	205	195	195
200	200	205	195	195

$\frac{1}{9}\times$

1	1	1
1	1	1
1	1	1

(200+200+200+200)/9	(200+200+205+200+200+205)/9	…	…	…
(200+200+200+200+195+195)/9	(200+200+205+200+200+205+195+195+200)/9	…	…	…
…	---	…	…	…
…	…	…	…	…
…	…	…	…	…

Actual image pixels

Extended pixels

Simplified Boundary Treatment

- Filtered image size=original image size
- Only compute in the valid region.
  - Assign 0 or keep original value in the inner boundary

	200	205	203	
	195	200	200	
	200	205	195	

$\frac{1}{9}\times$

1	1	1
1	1	1
1	1	1

Outer boundary: not considered

	200	205	203	
	195	…	200	
	200	205	195	

Inner boundary: not processed

Example: Simplified Boundary Treatment

100	100	100	100	100
100	200	205	203	100
100	195	200	200	100
100	200	205	195	100
100	100	100	100	100

$\frac{1}{9}\times$

1	1	1
1	1	1
1	1	1

$\frac{1}{16}\times$

1	2	1
2	4	2
1	2	1

100	100	100	100	100
100	144	167	145	100
100	167	200	168	100
100	144	166	144	100
100	100	100	100	100

100	100	100	100	100
100	156	176	158	100
100	174	201	175	100
100	156	175	156	100
100	100	100	100	100

Sample Matlab Program (With Simplified Boundary Treatment)

```
% readin bmp file
x = imread('lena.bmp');
[xh xw] = size(x);
y = double(x);
```

```
% define 2D filter
h = ones(5,5)/25;
[hh hw] = size(h);
hhh = (hh - 1) / 2;
hhw = (hw- 1) / 2;
```

```
% linear convolution, assuming the filter is non-separable (although this example filter is separable)
z = y; %or z=zeros(xh,xw) if not low-pass filter
for m = hhh + 1:xh - hhh,
            %skip first and last hhh rows to avoid boundary problems
   for n = hhw + 1:xw - hhw,
            %skip first and last hhw columns to avoid boundary problems
      tmpy = 0;
      for k = -hhh:hhh,
         for l = -hhw:hhw,
            tmpv = tmpv + y(m - k,n – l)* h(k + hhh + 1, l + hhw + 1);
                        %h(0,0) is stored in h(hhh+1,hhw+1)
         end
      end
      z(m, n) = tmpv;
%for more efficient matlab coding, you can replace the above loop with
            z(m,n)=sum(sum(y(m-hhh:m+hhh,n-hhw:n+hhw).*h))
   end
end
```

Separable Filters

- A filter is separable if $h(m, n)=h_x(m)h_y(n)$.
- Matrix representation

$$H = h_x h_y^T$$

  - Where h_x and h_y are column vectors
- Example

$$H_x = \frac{1}{4}\begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}; \quad H_y = \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

Separable Filtering

- If h(m,n) is separable, the 2D convolution can be accomplished by first applying 1D filtering along each row using $h_y(n)$, and then applying 1D filtering to the intermediate result along each column using the filter $h_x(n)$ (or column filtering followed by row filtering)
- Proof

$$
\begin{aligned}
f(m,n)*h(m,n) &= \sum_k \sum_l f(m-k,n-l)h_x(k)h_y(l) \\
&= \sum_k \left(\sum_l f(m-k,n-l)h_y(l) \right) h_x(k) \\
&= \sum_k g_y(m-k,n)h_x(k) \\
&= (f(m,n)*h_y(n))*h_x(m)
\end{aligned}
$$

Results Using Sobel Filters

Original image

Filtered image by H_x

Filtered image by H_y

$$H_x = \frac{1}{4}\begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}; \quad H_y = \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

What do Hx and Hy do?

Computation Cost: Non-Separable vs. Separable Filtering

- Suppose: Image size M*N, filter size K*L. Ignoring outer boundary pixels
- Non-separable filtering
  - Weighted average on each pixel: K*L mul; K*L – 1 add.
  - For all pixels: M*N*K*L mul; M*N*(K*L-1) add.
  - When M=N, K=L: M^2K^2 mul + $M^2(K^2-1)$ add.
- Separable filtering:
  - Each pixel in a row: L mul; L-1 add.
  - Each row: N*L mul; N*(L-1) add.
  - M rows: M*N*L mul; M*N*(L-1) add.
  - Each pixel in a column: K mul; K-1 add.
  - Each column: M*K mul; M*(K-1) add.
  - N columns: N*M*K mul; N*M*(K-1) add.
  - Total: M*N*(K+L) mul; M*N*(K+L-2) add.
  - When M=N, K=L: $2M^2K$ mul; $2M^2(K-1)$ add.
    - Significant savings if K (and L) is large!

MATLAB Function: conv2( )

- C=conv2(H,F,shape)
  - Shape=‘full’ (Default): C includes both outer and inner boundary, using zero padding
  - Shape=“same”: C includes the inner boundary, using zero padding
  - Shape=“valid’: C includes the convolved image without the inner boundary, computed without using pixels outside the original image
- C=conv2(h1,h2,F,shape)
  - Separable filtering with h1 for column filtering and h2 for row filtering

Notes about MATLAB implementation

- Input image needs to be converted to integer or double
  - Never do numerical operations on unsigned character type!
- Output image value may not be in the range of (0,255) and may not be integers
- To display or save the output image properly
  - Renormalize to (0,255) using a two-pass operation
    - First pass: save directly filtered value in an intermediate floating-point array
    - Second pass: find minimum and maximum values of the intermediate image, renormalize to (0,255) and rounding to integers
      - F= round((F1-fmin)*255/(fmax-fmin))
  - To display the unnormalized image directly in MATLAB, use imagesc(img). Or imshow(img, [ ]).

Convolution Theorem

- Convolution Theorem

$$f * h \quad \Leftrightarrow \quad F \times H, \quad f \times h \quad \Leftrightarrow \quad F * H$$

- Proof

$$g(m,n) = f(m,n) * h(m,n) = \sum_k \sum_l f(m-k, n-l) h(k,l)$$

FT on both sides

$$\begin{aligned}
G(u,v) &= \sum_{m,n} \sum_{k,l} f(m-k, n-l) h(k,l) e^{-j2\pi(mu+nv)} \\
&= \sum_{m,n} \sum_{k,l} f(m-k, n-l) e^{-j2\pi((m-k)u+(n-l)v)} h(k,l) e^{-j2\pi(ku+lv)} \\
&= \sum_{m,n} f(m-k, n-l) e^{-j2\pi((m-k)u+(n-l)v)} \sum_{k,l} h(k,l) e^{-j2\pi(ku+lv)} \\
&= \sum_{m',n'} f(m',n') e^{-j2\pi(m'u+n'v)} \sum_{k,l} h(k,l) e^{-j2\pi(ku+lv)} \\
&= F(u,v) \times H(u,v)
\end{aligned}$$

Another view of convolution theorem

$$f(m,n) * h(m,n) \quad \Leftrightarrow \quad F(u,v)H(u,v)$$

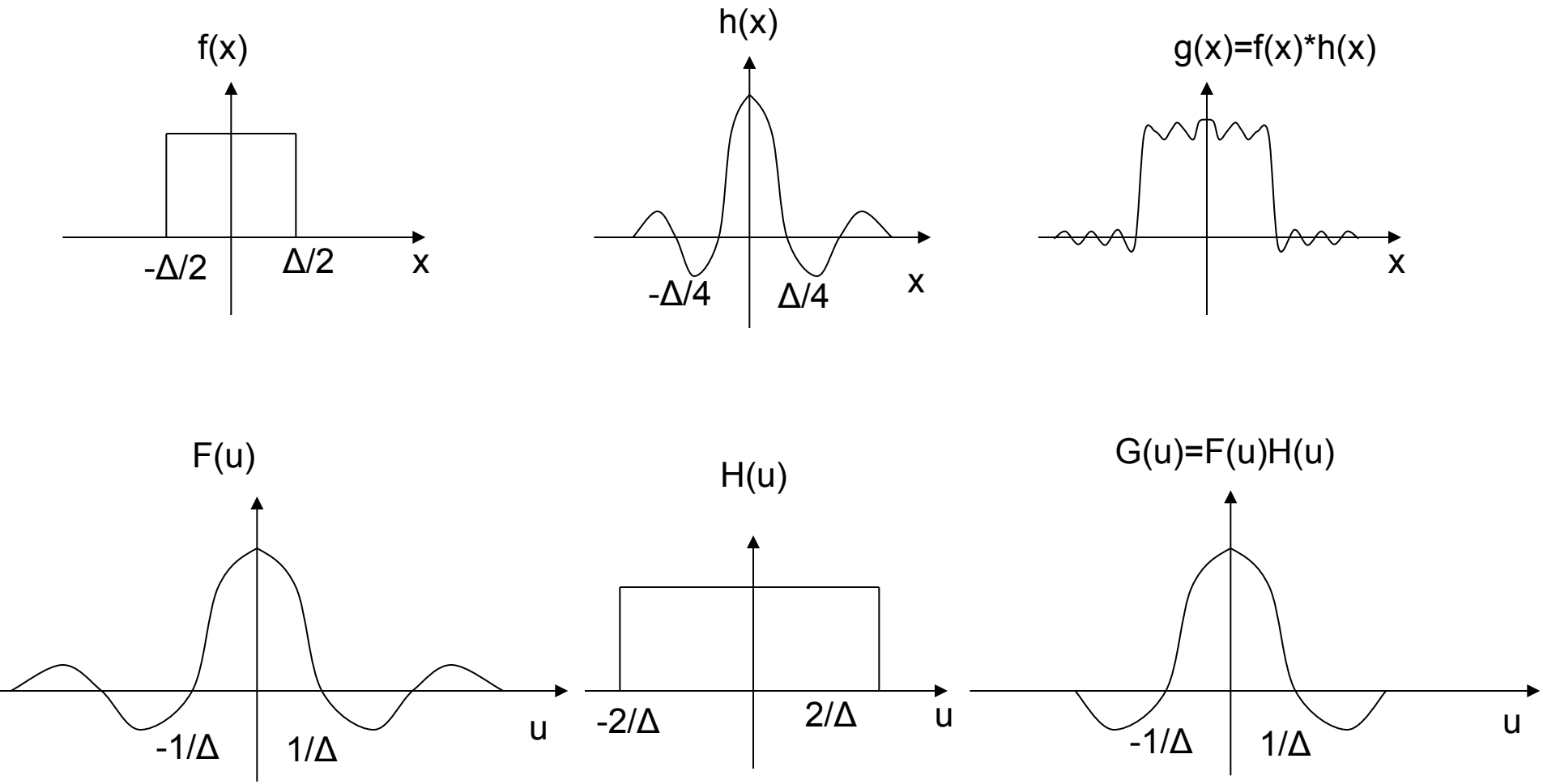
- F(u,v)H(u,v) = Modifying the signal's each frequency component' complex magnitude F(u,v) by H(u,v)

$$f(m,n) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} F(u,v)e^{j2\pi(mu+nv)}\,du\,dv$$

$$g(m,n) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} F(u,v)H(u,v)e^{j2\pi(mu+nv)}\,du\,dv$$

- H(u,v) is also called Frequency Response of the 2D LSI system
  - exp{2π (um+vn)} -> H(u,v) exp{2π (um+vn)} = |H(u,v)|exp{2π (um+vn)+P(H(u,v))}
  - Sinusoid (or complex exponential) input -> sinusoid (complex exponential) output!
  - H(u,v) describes how the magnitude and phase of a sinusoid input with frequency (u,v) are changed!

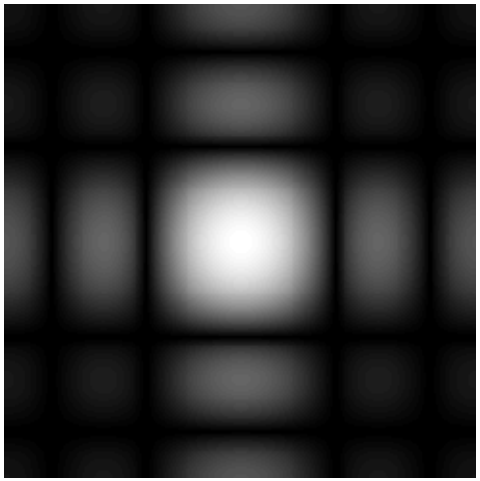
Explanation of Convolution in the Frequency Domain



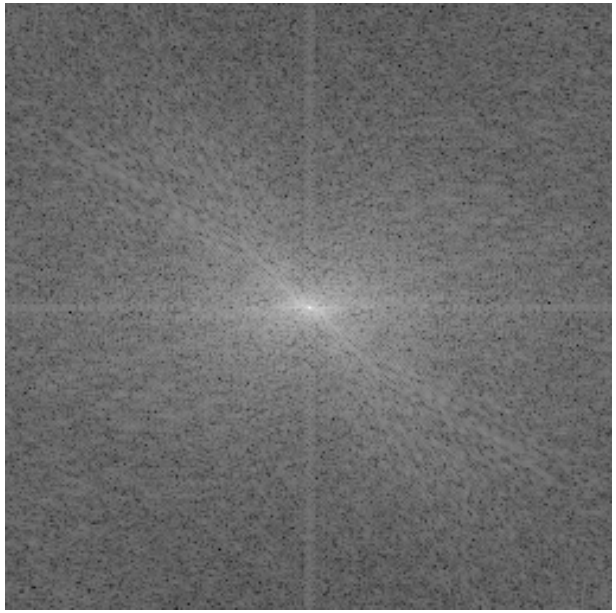
Example

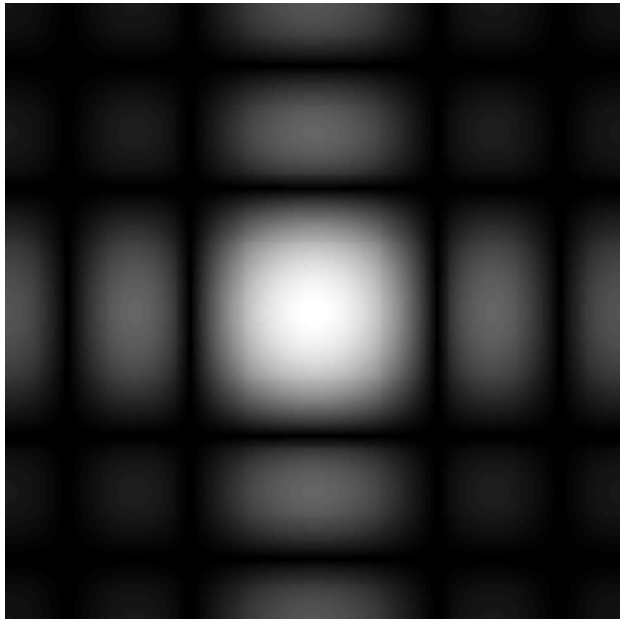
- Given a 2D filter, determine its frequency response. Apply to a given image, show original image and filtered image in pixel and freq. domain

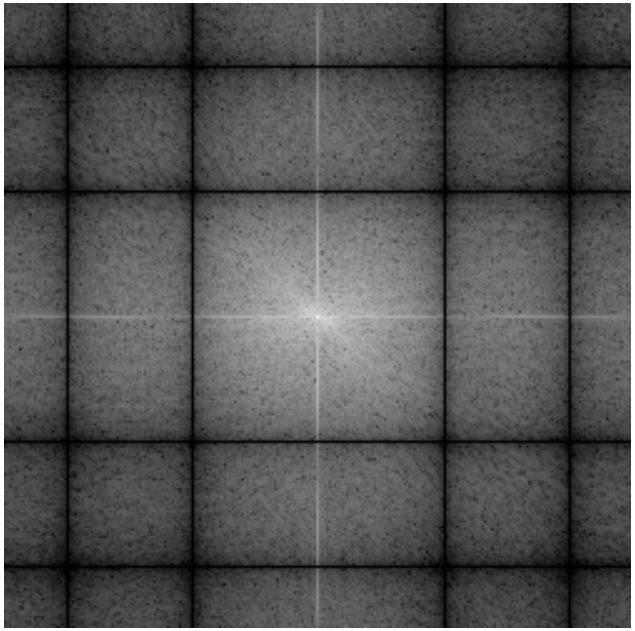
$$h = \frac{1}{25}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



$$h = \frac{1}{25}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



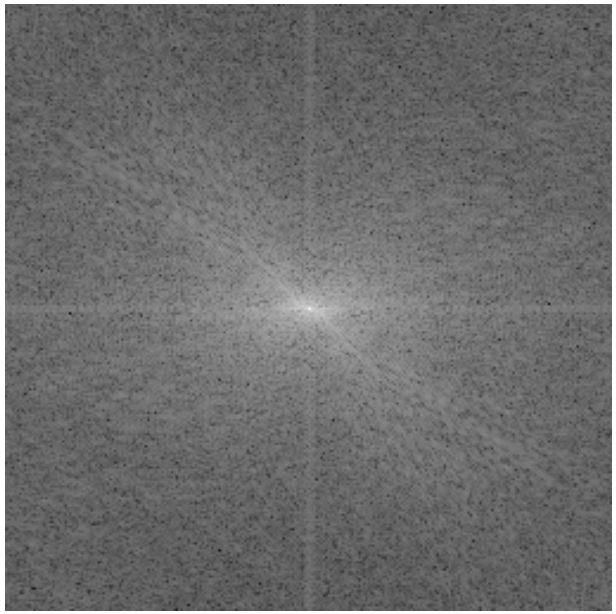


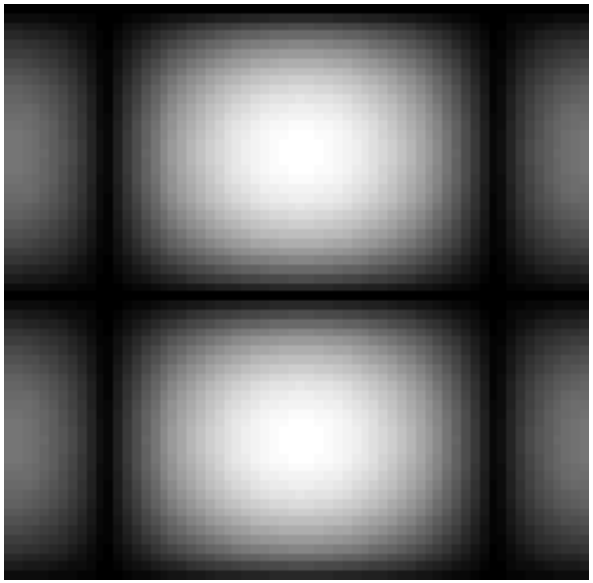


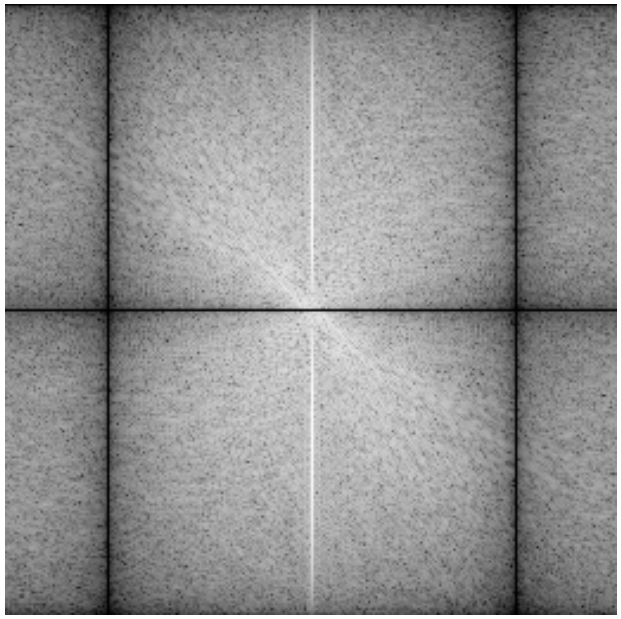
Matlab Program Used

```
x = imread('lena256.bmp');
figure(1); imshow(x);
f = double(x);
ff=abs(fft2(f));
figure(2); imagesc(fftshift(log(ff+1))); colormap(gray);truesize;axis off;
h = ones(5,5)/9;
hf=abs(freqz2(h));
figure(3);imagesc((log(hf+1)));colormap(gray);truesize;axis off;
y = conv2(f, h);
figure(4);imagesc(y);colormap(gray);truesize;axis off;
yf=abs(fft2(y));
figure(5);imagesc(fftshift(log(yf+1)));colormap(gray);truesize;axis off;
```

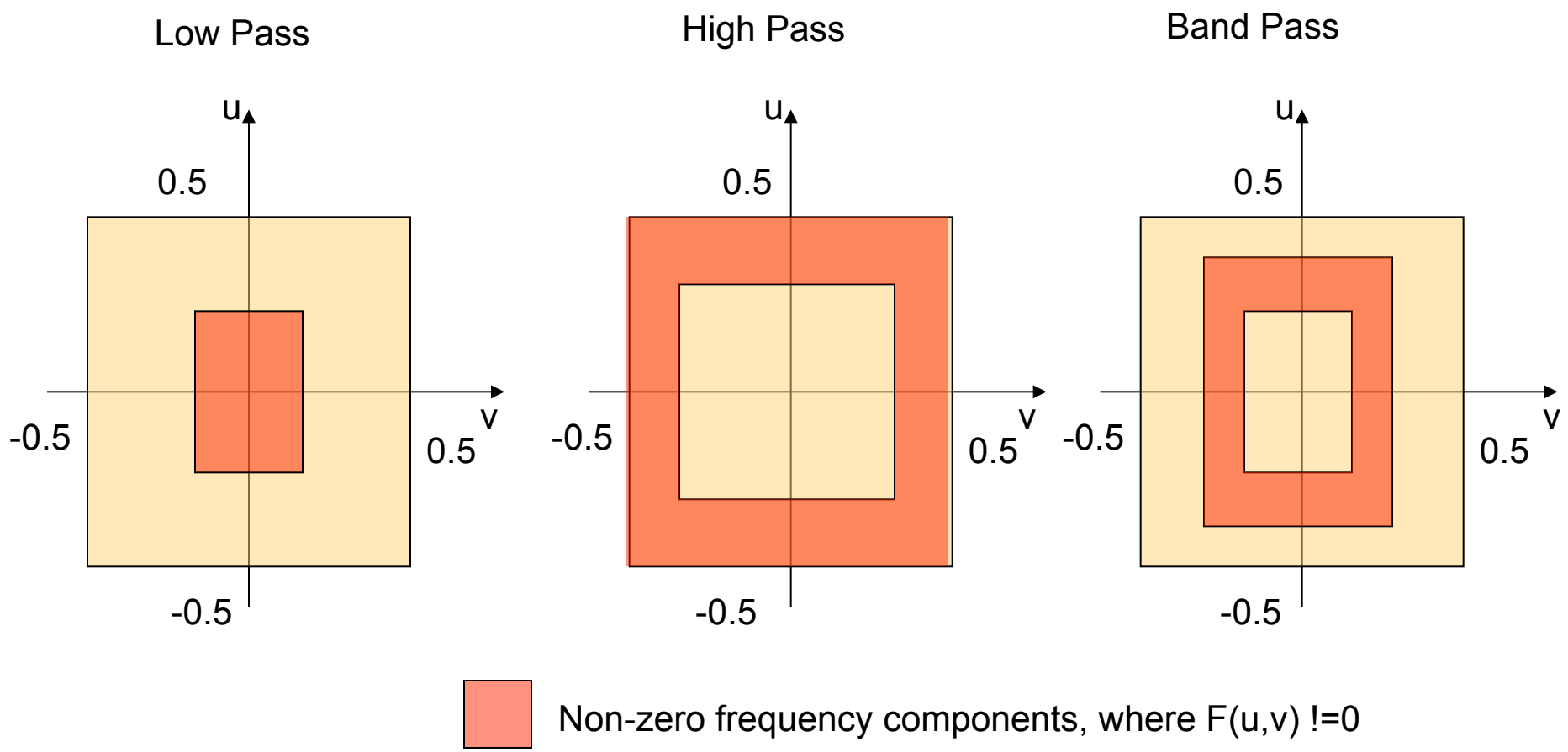
$$H_1 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$





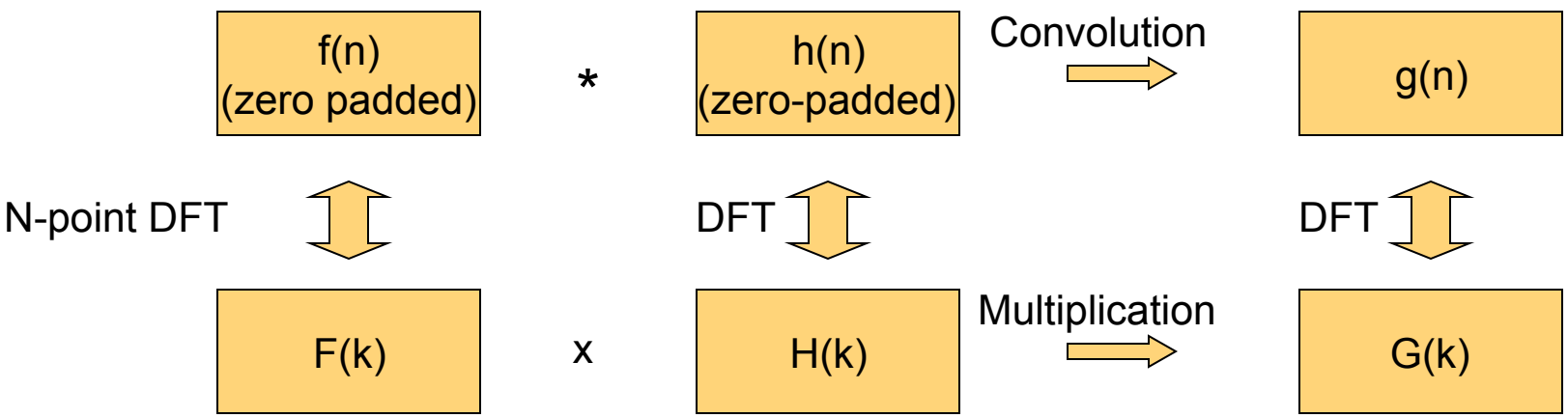


Typical Filter Types



Calculate Linear Convolution Using DFT

- 1D case
  - f(n) is length N_1, h(n) is length N_2
  - g(n) = f(n)*h(h) is length $N = N_1+N_2-1$.
  - To use DFT, need to **extend** f(n) and h(n) to length N by zero padding.
  - H(k) can be precalculated



Computing 2D Convolution Using 2D DFT

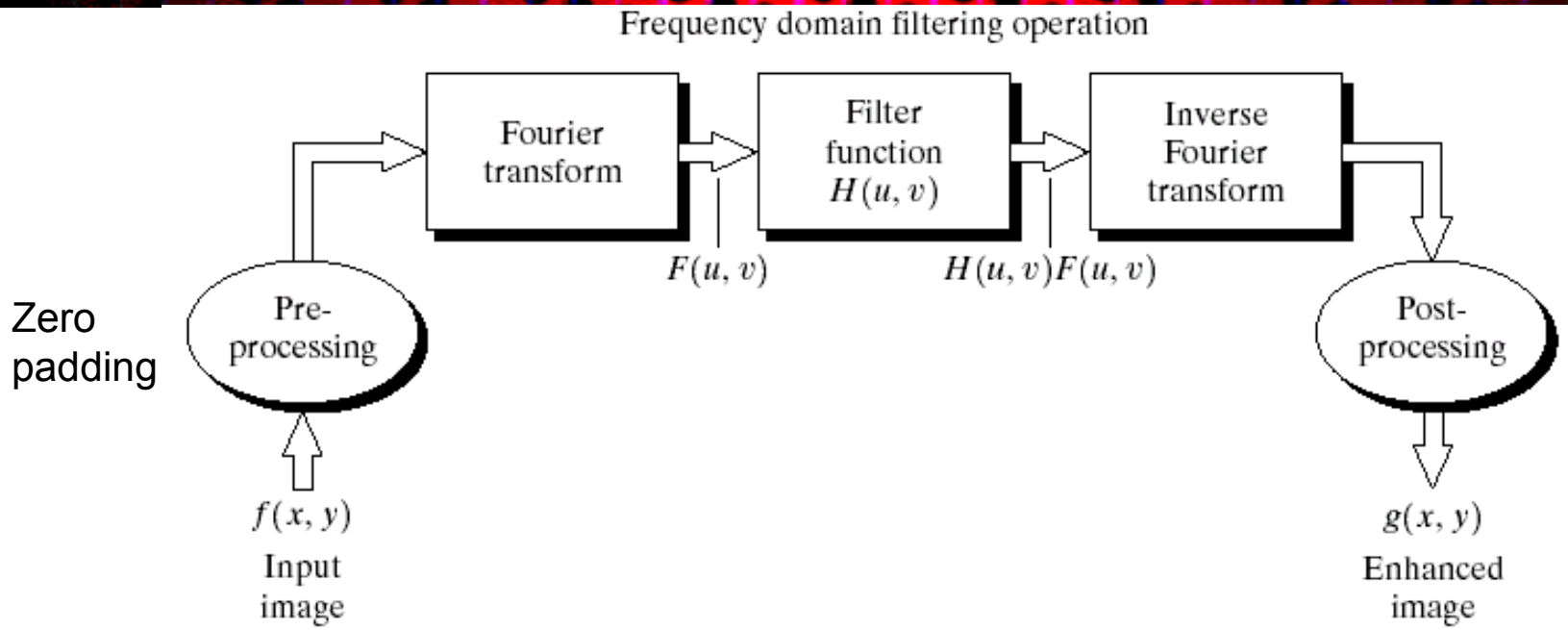


FIGURE 4.5 Basic steps for filtering in the frequency domain.

Relation between spatial and frequency domain operation:

$$g(x,y) = h(x,y) \otimes f(x,y) \quad \Leftrightarrow \quad G(u,v) = H(u,v)F(u,v)$$

$$h(x,y) = IDFT(H(u,v)), \quad H(u,v) = DFT(h(x,y)).$$

Typically DFT size=image size. This corresponds to circular convolution, which differs from linear convolution at the inner boundaries. Only correct in the valid region.

Image Filtering Using DFT

- Typically DFT size=image size. This corresponds to circular convolution, which differs from linear convolution at the inner boundaries. Only correct in the valid region.
- Circular convolution

$$f(n)\otimes h(n)=\sum_{k=0}^{N-1} f((n-k)\bmod(N))h(k)$$

$$f(n)\otimes h(n)\Leftrightarrow F_N(k)H_N(k)$$

- Image filters typically have short length to avoid boundary problems and ringing effect.
- For image filtering with short filters, it is more efficient to do convolution in the spatial domain.

Summary

- 2D CSFT and DSFT
  - Many properties of 1D CTFT and DTFT carry over, but there are a few things unique to 2D
  - Meaning of spatial frequency
  - 2D FT of separable signal = product of 1D FT
  - Rotation in space <-> rotation in frequency plane
- 2D linear convolution = weighted average of neighboring pixels
  - Filter=Point spread function (impulse response in 2D)
  - Any LSI (linear and shift invariant) operation can be represented by 2D convolution
  - DSFT of filter = frequency response = response to complex exponential input
- Computation of convolution:
  - boundary treatment, separable filtering
- Convolution theorem
  - Computation of convolution using DFT
- MATLAB function: conv2( ), freqz2( )

Reading Assignments

- Fourier transform and convolution:
  - Wang et al, Digital video processing and communications. Sec. 2.1, 2.2 (for general multi-dimensional case)
  - Optional: Woods, Multidimensional signal, image, and video processing and coding, Chap 1.
  - Good review of 1D FT and convolution:
  - Gonzalez & Woods, "Digital Image Processing", Prentice Hall, 3rd Ed, Chap 4. (You can skip sections 4.3, 4.8-4.10) Note: in Gonzalez & Woods, DTFT and DSFT are not introduced, the authors go directly to DFT.

Written Assignment (1)

1. Let $f(x,y) = \sin 2\pi f_0(x+y), \quad h(x,y) = \dfrac{\sin(2\pi f_c x)\sin(2\pi f_c y)}{\pi^2 xy}$

   Find the convolved signal $g(x, y) = f(x, y) * h(x, y)$ for the following two cases:

   a) $f_0/2 < f_c < f_0$; and b) $f_0 < f_c < 2f_0$.

   Hint: do the filtering in the frequency domain. Explain what happened by sketching the original signal, the filter, the convolution process and the convolved signal in the frequency domain.

2. Repeat the previous problem for

$$h(x,y) = \begin{cases} f_c^2, & -\dfrac{1}{2f_c} < \{x,y\} < \dfrac{1}{2f_c} \\ 0, & otherwise \end{cases}$$

3. Prove the rotation property of 2D CSFT
4. Show that the DSFT of f(m,n)=δ(m-n) is F(u,v)=δ(u+v)

Written Assignment (2)

5. For the three filters given below (assuming the origin is at the center):

a) find their Fourier transforms (2D DSFT);

b) sketch the magnitudes of the Fourier transforms. You should sketch by hand the DTFT as a function of u, when v=0 and when v=1/2; also as a function of v, when u=0 or ½. Also please plot the magnitude of DSFT as a function of both u and v, using Matlab plotting function.

c) Also use MATLAB freqz2( ) to compute and plot the magnitude of DSFT and compare to your answer.

c) Comment on the functions of the three filters.

In your calculation, you should make use of the separable property of the filter whenever appropriate. When the filter is not separable, you may be able to split the filter into several additive terms such that each term can be calculated more efficiently.

$$H_1 = \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \qquad H_2 = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}; \qquad H_3 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

MATLAB Assignment (1)

1. Write a Matlab program for implementing filtering of a gray scale image. Your program should allow you to specify the filter with an arbitrary size (but for simplicity, you can assume the filter size is KxL where both K and L are odd numbers, and the filter origin is at the center. Your program should read in a gray scale image, perform the filtering, display the original and filtered image, and save the filtered image into another file. You should write a separate function for the convolution that can be called by your main program. For simplicity, you can use the simplified boundary treatment. You should properly normalize the filtered image so that the resulting image values can be saved as 8-bit unsigned characters. Apply the filters given in the previous problem to a test image. Observe on the effect of these filters on your image. Note: you cannot use the MATLAB conv2( ) function. In your report, include your MATLAB code, the original test image and the images obtained with the three filters. Write down your observation of the effect of the filters.
2. Write a Matlab to simulate noise removal. First create a noisy image, by adding zero mean Gaussian random noise to your image using "imnoise()". You can specify the noise variance in "imnoise( )"). Then apply an averaging filter to the noise added image. For a chosen variance of the added noise, you need to try different window sizes (from 3x3 to 9x9) to see which one gives you the best trade-off between noise removal and blurring. Hand in your program, the original noise-added images at two different noise levels (0.01 and 0.1) and the corresponding filtered images with the best window sizes. Write down your observation. For the filtering operation, if your program in Prob. 1 does not work well, you could use the matlab "conv2()" function. Your program should allow the user to specify the window size as an input parameter.

MATLAB Assignment (2)

3. (Optional: no extra credits, purely for your curiosity) Write a convolution program assuming the filter is separable. Your program should allow you to specify the horizontal and vertical filters, and call a 1D convolution sub-program to accomplish the 2D convolution. Note: you cannot use the MATLAB conv( ) or conv2() function.