Solution to Chap. 13

**Solution 13.2** The functions of RF coils are twofold. (1) During radio-frequency excitation, a relatively large amount of current is generated in the coil using an RF amplifier (with a power requirement of approximately 2 kW for human imaging). Ideally, this coil then produces a relatively uniform $B_1$ field throughout the entire imaging volume in order that the same tip angle is generated in each isocromat in the volume. (2) On reception, an RF coil must pick up very low amplitude magnetic fields, which produce very small currents in the coil. Transmission and receiver RF coils can be the same coil, but are different when high SNR or fast imaging is required.

**ENCODING SPATIAL POSITION AND MR IMAGING EQUATION**

**Solution 13.3**

(a)

\[ \Delta \nu = \gamma G_c \Delta z = 4.258 \text{ kHz} \times 1 \text{ G/mm} \times 10 \text{ mm} = 42.58 \text{ kHz}. \]

(b)

\[ B_1(t) = A \Delta \nu \sin(\Delta \nu t) e^{j2\pi \Delta \nu t}. \]

where $\Delta \nu = \gamma G_c + \gamma B_0 = 212.9 \text{ kHz} + \gamma B_0$. Therefore, we have:

\[ B_1(t) = A \times 42.58 \text{ kHz} e^{j2\pi \Delta \nu t}. \]

In rotating frame,

\[ B_1(t) = A \times 42.58 \text{ kHz} e^{j2\pi \gamma B_0 t}, \]

where $A$ depends on the tip angle.

**Solution 13.4**

(a) Start with the Fourier transform pair

\[ \mathcal{F}\{e^{-\pi t^2}\} = e^{-\pi u^2}. \]

Then since

\[ A_6 \exp\left(-t^2/\sigma^2\right) = A_6 \exp\left\{ -\pi \left( \frac{t}{\sqrt{\pi} \sigma} \right)^2 \right\}, \]

we can use the Fourier scaling theorem to get

\[ \mathcal{F}\left\{ A_6 \exp\left(-t^2/\sigma^2\right) \right\} = A_6 \sqrt{\pi} \sigma \exp\left\{ -\pi^2 \sigma^2 u^2 \right\}. \]

The FWHM is found as follows:

\[
\begin{align*}
1/2 &= \exp\left\{ -\pi^2 \sigma^2 u^2 \right\} \\
\ln(1/2) &= -\pi^2 \sigma^2 u^2 \\
u^2 &= \frac{1}{\pi^2 \sigma^2} \ln 2 \\
\sigma^2 &= \frac{1}{\pi^2} \ln 2 = 265 \text{ Hz} \\
\text{FWHM} &= 2 \times 265 \text{ Hz} = 530 \text{ Hz}
\end{align*}
\]

This defines the frequency interval that is excited $\Delta \nu = 530 \text{ Hz}$, and using Equation (13.12) gives

\[ \Delta x = \frac{530 \text{ Hz}}{42.58 \times 10^6 \text{ Hz/T} \times 10^{-4} \text{ T/cm}} = 1.2 \text{ mm}. \]
(b) The new gradient strength is $G_z' = 0.5G_z$. So, the new slice thickness is

\[
\Delta z = \frac{\Delta \nu}{\gamma G_z'} = \frac{\Delta \nu}{\gamma 0.5G_z} = 2 \times \Delta z
\]

Halving the gradient strength doubles the slice thickness.

Now suppose that $\sigma' = 0.5\sigma$. Starting from the original (without using $G_z'$), we have from previous work that

\[
\nu' = \sqrt{\frac{1}{\pi^2 \sigma'^2 \ln 2}} = \frac{1}{\sigma'} = \frac{1}{0.5\sigma'} \sqrt{\frac{1}{\pi^2 \ln 2}} = 2\mu
\]

Therefore, the new frequency range $\Delta \nu'$ is double what it was before, which doubles the slice thickness. If used in combination, the slice thickness would be four times thicker.

(c) In this case, only the RF pulse is changed, so the slice thickness is doubled: $\Delta z' = 2.4$ mm.

(d) Doubling the slice thickness improves the SNR by a factor of two. The overall imaging time is slightly smaller (although this will not affect SNR since the actual ADC time is unaffected). Image resolution will be degraded in the through-plane direction.

**Solution 13.12**

(a) In this interval, the Fourier trajectory goes from the origin to the point $(-0.25, -0.5)$ mm$^{-1}$. Using the relations:

\[
u = \gamma G_x', \quad \nu = \gamma G_y'
\]

leads to

\[
G_x = \frac{\nu}{\gamma T_s} = \frac{-0.25}{4258 \text{ Hz/G 0.0001 sec}} = -0.587 \text{ G/mm}
\]

\[
G_y = \frac{\nu}{\gamma T_s} = \frac{-0.5}{4258 \text{ Hz/G 0.0001 sec}} = -1.174 \text{ G/mm}
\]

(b) A similar argument as in (a) leads to

\[
G_x = \frac{\Delta \nu}{\gamma T_s} = \frac{0.5}{4258 \text{ Hz/G 0.01 sec}} = 11.7 \text{ mG/mm}
\]

\[
G_y = \frac{\Delta \nu}{\gamma T_s} = \frac{1.0}{4258 \text{ Hz/G 0.01 sec}} = 23.5 \text{ mG/mm}
\]

The sampling rate is

\[
f_s = \frac{128}{10 \text{ ms}} = 12.8 \text{ kHz}.
\]
Solution 13.13 The required timing diagram is shown in Figure S13.4.

The timings and the amplitudes of the gradients:

\[ k_r = \gamma \int G_x dt, \quad k_y = \gamma \int G_y dt \]

\[ -1 \text{ mm}^{-1} = -1000 \text{ m}^{-1} = 42.6 \times 10^6 \times (-G_1) \times t_1 \Rightarrow G_1 \times t_1 = 23.5 \times 10^{-6} \left( \frac{T \cdot \text{sec}}{m} \right) \]

\[ 0.5 \text{ mm}^{-1} = 500 \text{ m}^{-1} = 42.6 \times 10^6 \times G_2 \times t_1 \Rightarrow G_2 \times t_1 = 11.7 \times 10^{-6} \left( \frac{T \cdot \text{sec}}{m} \right) \]

\[ 2 \text{ mm}^{-1} = 2000 \text{ m}^{-1} = 42.6 \times 10^6 \times G_1 \times t_2 \Rightarrow G_1 \times t_2 = 47 \times 10^{-6} \left( \frac{T \cdot \text{sec}}{m} \right) \]

\[ 0.1 \text{ mm}^{-1} = 100 \text{ m}^{-1} = 42.6 \times 10^6 \times G_3 \times t_3 \Rightarrow G_3 \times t_3 = 2.3 \times 10^{-6} \left( \frac{T \cdot \text{sec}}{m} \right) \]

\[ -2 \text{ mm}^{-1} = -2000 \text{ m}^{-1} = 42.6 \times 10^6 \times (-G_1) \times t_4 \Rightarrow G_1 \times t_4 = 47 \times 10^{-6} \left( \frac{T \cdot \text{sec}}{m} \right) \]

Now, let \( G_1 = 10 \text{ mT/m}, G_2 = 5 \text{ mT/m}, \) and \( G_3 = 1 \text{ mT/m}. \) Then, \( t_1 = 2.3 \text{ msec}, t_2 = 4.7 \text{ msec} \)
\( t_3 = 2.3 \text{ msec}, \) and \( t_4 = 4.7 \text{ msec}. \)