Solution to Chap.6

Solution 6.5 Since $f(x, y)$ is rotationally symmetric, $g(\ell, \theta) = g(\ell, 0)$. Hence,

$$\begin{aligned} g(\ell, \theta) &= \int_{L(\ell,\theta)} f(x, y) ds \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} \delta(x \cos(\theta) - \ell)\, dx dy \\ &= \int_{-\infty}^{\infty} e^{-\ell^2-y^2}\, dy \\ &= e^{-l^2} \int_{-\infty}^{\infty} e^{-y^2}\, dy\,. \end{aligned}$$

Since

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\, dx = 1,$$

then,

$$\int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}.$$

Thus,

$$g(\ell, \theta) = \sqrt{\pi} e^{-\ell^2}.$$

Solution 6.9 We start with the convolution integral

$$g(x, y) = \int_{\xi}\int_{\eta} f(\xi, \eta) h(x - \xi, y - \eta)\, d\xi d\eta\,.$$

Now, we perform the following steps:

$$\begin{aligned} \mathcal{R}\{g\} &= \int_y \int_x \int_\eta \int_\xi f(\xi, \eta) h(x - \xi, y - \eta)\, d\xi d\eta \delta(x \cos\theta + y \sin\theta - \ell)\, dx dy \\ &= \int_\eta \int_\xi f(\xi, \eta) \int_y \int_x h(x - \xi, y - \eta) \delta(x \cos\theta + y \sin\theta - \ell)\, dx dy\, d\xi d\eta \\ &= \int_\eta \int_\xi f(\xi, \eta) \int_{y'} \int_{x'} h(x', y') \delta(x' \cos\theta + y' \sin\theta - [\ell - \xi \cos\theta - \eta \sin\theta])\, dx' dy'\, d\xi d\eta \\ &= \int_\eta \int_\xi f(\xi, \eta) \mathcal{R}\{h\}(\ell - \xi \cos\theta - \eta \sin\theta, \theta)\, d\xi d\eta \\ &= \int_\eta \int_\xi f(\xi, \eta) \int_{\ell'} \mathcal{R}\{h\}(\ell - \ell', \theta) \delta(\xi \cos\theta + \eta \sin\theta - \ell')\, d\ell'\, d\xi d\eta \\ &= \int_{\ell'} \mathcal{R}\{h\}(\ell - \ell', \theta) \int_\eta \int_\xi f(\xi, \eta) \delta(\xi \cos\theta + \eta \sin\theta - \ell')\, d\xi d\eta\, d\ell' \\ &= \int_{\ell'} \mathcal{R}\{h\}(\ell - \ell', \theta) \mathcal{R}\{f\}(\ell', \theta) d\ell' \\ &= \mathcal{R}\{h\} * \mathcal{R}\{f\} \end{aligned}$$

which was to be proved.

Solution 6.10

(a)

$$\begin{aligned} g_s(\ell, \theta + \pi/2) &= \iint s(x, y)\delta(x\cos(\theta + \pi/2) + y\sin(\theta + \pi/2) - \ell)\,dxdy \\ &= \iint s(x, y)\delta(-x\sin\theta + y\cos\theta - \ell)\,dxdy \\ &= \iint s(-v, u)\delta(u\cos\theta + v\sin\theta - \ell)\,dudv\ (u = y, v = -x) \\ &= \iint s(u, v)\delta(u\cos\theta + v\sin\theta - \ell)\,dudv \\ &= g_s(\ell, \theta) \end{aligned}$$

(b)

$$\begin{aligned} g_s(\ell, -\theta) &= \iint s(x, y)\delta(x\cos(-\theta) + y\sin(-\theta) - \ell)\,dxdy \\ &= \iint s(x, y)\delta(x\cos\theta - y\sin\theta - \ell)\,dxdy \\ &= \iint s(u, -v)\delta(u\cos\theta + v\sin\theta - \ell)\,dudv\ (u = x, v = -y) \\ &= \iint s(u, v)\delta(u\cos\theta + v\sin\theta - \ell)\,dudv \\ &= g_s(\ell, \theta) \end{aligned}$$

(c) Let

$$\tilde{g}_s(\ell, \theta) = \begin{cases} g_s(\ell, \theta) & 0 \le \theta < \frac{\pi}{4} \\ g_s\left(\ell, \frac{\pi}{2} - \theta\right) & \frac{\pi}{4} \le \theta < \frac{\pi}{2} \end{cases},$$

which covers $0 \le \theta < \pi/2$. Then

$$g_s(\ell, \theta) = \begin{cases} \tilde{g}_s(\ell, \theta) & 0 \le \theta < \frac{\pi}{2} \\ \tilde{g}_s\left(\ell, \theta - \frac{\pi}{2}\right) & \frac{\pi}{2} \le \theta \le \pi \end{cases},$$

covers $0 \le \theta < \pi$.

(d) See Figure S6.1.

(e) See Figure S6.2.

From simple rotations, we have $\ell_1 = -\cos\theta - \sin\theta$, $\ell_2 = -\cos\theta + \sin\theta$, $\ell_3 = \cos\theta - \sin\theta$, and $\ell_4 = \cos\theta + \sin\theta$.

By similar triangles, we have:

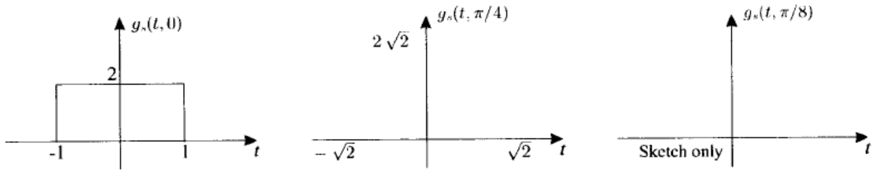

Figure S6.1 Sketch of projections at different angles. [Problem 6.10(d)]

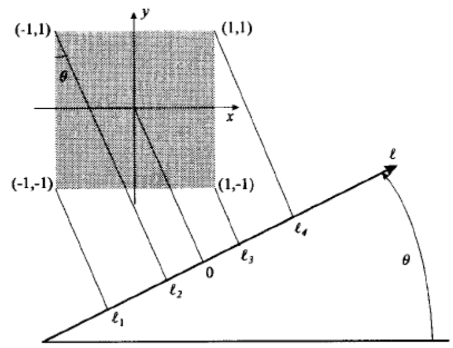

Figure S6.2 [Problem 6.10(e)]

$$\underline{\ell_1 \le \ell \le \ell_2:} \quad \frac{\ell - \ell_1}{\ell_2 - \ell_1} = \frac{g(\ell,\theta)}{\left(\frac{2}{\cos\theta}\right)}$$

$$\Rightarrow g(\ell,\theta) = \frac{2}{\cos\theta}\left(\frac{\ell + \cos\theta + \sin\theta}{2\sin\theta}\right)$$

$$= \frac{\ell + \cos\theta + \sin\theta}{\cos\theta\sin\theta}$$

$$\underline{\ell_2 \le \ell \le \ell_3:} \quad g(\ell,\theta) = \frac{2}{\cos\theta}$$

$$\underline{\ell_3 \le \ell \le \ell_4:} \quad \frac{\ell_4 - \ell}{\ell_4 - \ell_3} = \frac{g(\ell,\theta)}{\left(\frac{2}{\cos\theta}\right)}$$

$$\Rightarrow g(\ell,\theta) = \frac{\cos\theta + \sin\theta - \ell}{\cos\theta\sin\theta}.$$

and $g(\ell,\theta) = 0$ elsewhere.

Solution 6.13

(a)

$$g(l,60) = \begin{cases} \sqrt{3}\mu(a/2+l) & -a/2 \le l \le 0 \\ \sqrt{3}\mu(a/2-l) & 0 \le l \le a/2 \\ 0 & otherwise \end{cases}$$

The projection $g(\ell, 60°)$ is shown in Figure S6.9.

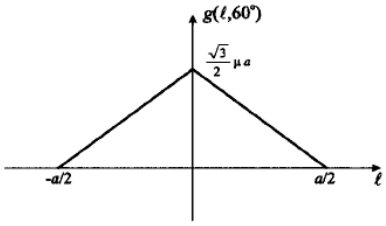

Figure S6.9 [Problem 6.13(a)]

(b)

$$\begin{aligned} b_{60°}\left(0, \frac{a}{4}\right) &= g\left(0\cos 60° + \frac{a}{4}\sin 60°, 60°\right) \\ &= g\left(\frac{\sqrt{3}a}{8}, 60°\right) \\ &= \sqrt{3}\mu\left(\frac{a}{2} - \frac{\sqrt{3}a}{8}\right) \\ &= \frac{\sqrt{3}\mu a}{8}(4 - \sqrt{3}). \end{aligned}$$

(c) Let function $f(t)$ be defined by scaling a rect function as:

$$f(t) = \begin{cases} K, & -\frac{p}{2} \le t \le \frac{p}{2} \\ 0, & \text{otherwise} \end{cases} .$$

The convolution $f(t) * f(t)$ is given by:

$$f(t) * f(t) = \begin{cases} K^2(p+t), & -p \le t \le 0 \\ K^2(p-t), & 0 \le t \le p \\ 0, & \text{otherwise} \end{cases} .$$

Comparing with $g(\ell, 60°)$, we see that it is a convolution of function $f(\ell)$ with itself, where $K^2 = \sqrt{3}\mu$, and $p = a/2$:

$$g(\ell, 60°) = \sqrt{\sqrt{3}\mu}\,\mathrm{rect}\left(\frac{2\ell}{a}\right) * \sqrt{\sqrt{3}\mu}\,\mathrm{rect}\left(\frac{2\ell}{a}\right).$$

By the projection slice theorem, we get $F(\varrho\cos\theta, \varrho\sin\theta) = G(\varrho, \theta) = \mathcal{F}\{g(\ell, \theta)\}$. Since $g(\ell, 60°)$ is expressed as a convolution of a rect function with itself, we have:

$$F\{g(l, 60^\circ)\} = [F\{\sqrt{\sqrt{3}\mu} \cdot rect(\frac{2l}{a})\}]^2$$

$$= \frac{\sqrt{3}\mu a^2}{4}\,\mathrm{sinc}^2\left(\frac{a\varrho}{2}\right).$$

Therefore,

$$F(\varrho\cos 60°, \varrho\sin 60°) = \frac{\sqrt{3}\mu a^2}{4}\,\mathrm{sinc}^2\left(\frac{a\varrho}{2}\right).$$

Solution 6.17

(a) $b_\theta(x, y) = g(x\cos\theta + y\sin\theta, \theta)$.

(b)

$$\begin{aligned}\ell &= x\cos\theta + y\sin\theta = 1\cos 30^\circ + 2\sin 30^\circ \\ &= 0.866 + 1 = 1.866.\end{aligned}$$

$$b_{30^\circ}(1, 2) = g(1.866, 30^\circ) \approx 0.155.$$

(c) No, because $g(\ell, 30^\circ)$ does not say anything about $g(\ell, 45^\circ)$.

(d) $210^\circ = 30^\circ + 180^\circ$. Thus, this is the "opposite" projection, and

$$g(\ell, 210^\circ = g(-\ell, 30^\circ) = 0.155.$$

(e) See Figure S6.10. The image always has the same value along the lines with a slope of $\tan 120^\circ = -\sqrt{3}/3$.

(f) No, because to determine $b_{30^\circ}(1, 2)$, we need $\ell = 1.866$ as shown in (b), which is not an integer. An approximate value might be to choose $\ell = 2$, which yields 0.135.

(g) $\ell = 2 \times 0.866 + 1 \times 0.5 = 2.232$, which is again not an integer. Thus, the exact value still cannot be determined.
For approximation, still $\ell = 2$, and the approximate value is still 0.135.

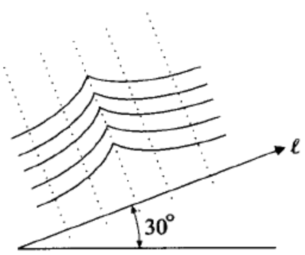

Figure S6.10 [Problem 6.17(e)]

Solution 6.19 Beam width W has the effect of convolving projection with rect $\left(\frac{\ell}{W}\right)$.

Ignoring sampling (at first) and pretending that we don't know about the distortion, CBP yields

$$\tilde{f}(x, y) = \int_0^\pi \left[(g_\theta(\ell) * \text{rect}\left(\frac{\ell}{W}\right) * c(\ell)\right]_{\ell = x\cos\theta + y\sin\theta} d\theta$$

or

$$\tilde{f}(x, y) = \int_0^\pi \int_{-\infty}^\infty \left[(g_\theta(\ell) * \text{rect}\left(\frac{\ell}{W}\right) * c(\ell)\right] \delta(x\cos\theta + y\sin\theta - \ell)\, d\ell d\theta.$$

We let $g_\theta(\ell) = \delta(\ell) =$ 2-D Radon transform of $\delta(x, y)$ to find the impulse response. But $\delta(\ell) * \text{rect}\left(\frac{\ell}{W}\right) * c(\ell) = \text{rect}\left(\frac{\ell}{W}\right) * c(\ell)$. Therefore, the impulse response in the inverse 2-D Radon transform of $g_\theta(\ell) = \text{rect}\left(\frac{\ell}{W}\right)$, or the function $h(x, y)$ whose 2-D Radon transform is rect $\left(\frac{\ell}{W}\right)$. The function has support on the disk with diameter W centered at the origin, but is not constant within, as shown in Figure S6.11.

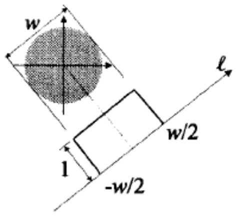

Figure S6.11 [Problem 6.19]

The easiest way to determine $h(x, y)$ is via the projection-slice theorem.

$$\mathcal{F}\left\{\text{rect}\left(\frac{\ell}{W}\right)\right\} = |W|\,\text{sinc}(W\varrho)$$

and since all projections are the same, we conclude that

$$H(\varrho) = |W|\,\text{sinc}(W\varrho) = |W|\frac{\sin(\pi W\varrho)}{\pi W\varrho}$$

$H(\varrho)$ is the radial part of $\mathcal{F}(h(x, y))$. The inverse transform of $H(\varrho)$ also has circular symmetry and is given by the inverse Hankel transform (Macovski, page 16, for example):

$$h(r) = 2\pi\int_0^\infty H(\varrho)J_0(2\pi\varrho r)\varrho\,d\varrho$$

where J_0 is the Bessel function of order 0.

From tables (Bracewell for example) we find

$$\frac{\text{rect}\left(\frac{r}{2a}\right)}{\sqrt{a^2 - r^2}} \longleftrightarrow \frac{\sin(2\pi a\varrho)}{\varrho}.$$

Hence

$$h(r) = \left(\frac{1}{\pi W}\right)\frac{\text{rect}\left(\frac{r}{W}\right)}{\sqrt{\left(\frac{W}{2}\right)^2 - r^2}}$$

which gives

$$h(x, y) = \left(\frac{1}{\pi W}\right)\frac{\text{rect}\left(\frac{\sqrt{x^2+y^2}}{W}\right)}{\sqrt{\left(\frac{W}{2}\right)^2 - (x^2+y^2)}}$$

Finally, we conclude that

$$\tilde{f}(x, y) = f(x, y) * h(x, y).$$

Additional comments: $h(x, y)$ is a low-pass filter since its Fourier transform decays as a sinc in ϱ. Hence, $\tilde{f}(x, y)$ is a blurred version of $f(x, y)$ as expected. However, $h(x, y)$ has finite support, so that the blurring is strictly local-in fact contributions occur only from over the disk of radius $\frac{W}{2}$.

But h(x,y) approaches ∞ asymptotically at $r = \frac{W}{2}$, which means that the contribution to blurring at exactly the radius $\frac{W}{2}$ can be very strong and one might expect to see circular artifacts of radius $\frac{W}{2}$ near bright point objects.

In a real system CBP is done for sampled data. The convolution $g_\theta(l) * rec(\frac{l}{w})$ **is continuous convolution, however, followed by discrete sampling. Therefore we might write**

$$\tilde{f}(x,y) = \frac{\pi}{M}\sum_{j=1}^{M} T \sum_{i=1}^{N} [g_\theta(s) * rec(\frac{s}{w})]_{x=iT} \; C(x\cos\theta_j + y\sin\theta_j - iT)$$

But there is not more we can say analytically about $\hat{f}(x, y)$ vs. $f(x, y)$.

Note: h(x,y)=H^{-1} {s(ρ)}.

Solution 6.20 Assume a rectangularly windowed rho-filter is used. The SNR can be computed using Eq. (6.74). By assumption, $M = 100$, $C = 0.05$, $\bar{\mu} = 0.15\ \text{cm}^{-1}$. Since the detectors are touching each other, $k = 1$. Since the cylinder has a diameter of 20cm, and the detector dimension is 2.0mm ×2.0mm, the number of measurements per projection is

$$\frac{20\ \text{cm}}{2\ \text{mm}} = 100.$$

Hence,

$$0.1\text{R/projection} \times \frac{1}{100}\text{projection/measurement} = 0.001\text{R/measurement}.$$

The worst-case intersection length of a beam with the water is 20cm. Therefore, the worst-case $\bar{N}$ is

$$\begin{aligned}\bar{N} &= 2.5 \times 10^{10}\frac{\text{photons}}{\text{cm}^2\text{R}} \times 0.04\ \text{cm}^2 \times 0.001\frac{\text{R}}{\text{measurement}} e^{-0.15\times 10} \\ &\approx 50 \times 10^3 \text{photons/measurement}.\end{aligned}$$

Thus,

$$\begin{aligned}\text{SNR} &\approx 0.4kC\bar{\mu}\sqrt{\bar{N}Mw} \\ &= 0.4 \times 1 \times 0.05 \times 0.15\ \text{cm}^{-1}\sqrt{50 \times 10^3 \times 100} \times 0.2 \\ &\approx 1.3\end{aligned}$$

Also, $\text{SNR} = 20\log_{10} 1.3 \approx 2.5$ dB, since the SNR is not a power ratio as defined.

Consider a 4x4 image that contains a diagonal line

I=[0,0,0,1;0,0,1,0;0,1,0,0;1,0,0,0];

- *a) determine its projections in the directions: 0, 45,90,135 degrees.*
- *b) determine the backprojected image from each projection;*
- *c) determine the reconstructed images by using projections in the 0 and 90 degrees only.*
- *d) determine the reconstructed images by using all projections. Comment on the difference from c).*

Solution:

Original image:

$$I = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(a) projections in directions:

0 degree: 1, 1, 1, 1

45 degree: 0, 0, 0, 4, 0, 0, 0

90 degree: 1, 1, 1, 1

135 degree: 1, 0, 1, 0, 1, 0, 1

(b) backprojections in each direction:

Each projection should be normalized by the number of pixels along the projection path.

$$b_{0^\circ} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad b_{45^\circ} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$b_{90^\circ} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad b_{135^\circ} = \begin{pmatrix} 0 & 1/3 & 0 & 1 \\ 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 \\ 1 & 0 & 1/3 & 0 \end{pmatrix}$$

(c) $$b_{0^\circ} + b_{90^\circ} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

(d) $$b_{0^\circ} + b_{45^\circ} + b_{90^\circ} + b_{135^\circ} = \begin{pmatrix} 1/2 & 5/6 & 1/2 & 5/2 \\ 5/6 & 1/2 & 11/6 & 1/2 \\ 1/2 & 11/6 & 1/2 & 5/6 \\ 5/2 & 1/2 & 5/6 & 1/2 \end{pmatrix}$$

Using all 4 backprojections, reconstructed image is better to reflect the original image than (c) using 2 backprojections.