Median Filtering and Morphological Filtering

Yao Wang
Polytechnic University, Brooklyn, NY 11201

With contribution from Zhu Liu, Onur Guleryuz, and Gonzalez/Woods, Digital Image Processing, 2ed
Lecture Outline

- Median filter
- Rank order filter
- Bilevel Morphological filters
  - Dilation and erosion
  - Opening and closing
- Grayscale Morphological filters
Median Filter

• Problem with Averaging Filter
  – Blur edges and details in an image
  – Not effective for impulse noise (Salt-and-pepper)

• Median filter:
  – Taking the median value instead of the average or weighted average of pixels in the window
    • Sort all the pixels in an increasing order, take the middle one
  – The window shape does not need to be a square
  – Special shapes can preserve line structures
Median Filter: 3x3 Square Window

Window shape

Matlab command: medfilt2(A,[3 3])
Median Filter: 3x3 Cross Window

Note that the edges of the center square are better reserved.
Example

**FIGURE 3.37** (a) X-ray image of circuit board corrupted by salt-and-pepper noise. (b) Noise reduction with a $3 \times 3$ averaging mask. (c) Noise reduction with a $3 \times 3$ median filter. (Original image courtesy of Mr. Joseph E. Pascente, Lixi, Inc.)
Rank order filters

• Rank order filters
  – Instead of taking the mean, rank all pixel values in the window, take the n-th order value.
  – E.g. max or min or median

• Properties
  – Non-linear $T(f_1 + f_2) \neq T(f_1) + T(f_2)$
Multi-level Median Filtering

- To reduce the computation, one can concatenate several small median filters to realize a large window operation.
- When the small windows are designed properly, this approach can also help reserve edges better.
Hybrid Linear/Median Filter

- One can combine median filters with linear or rank order filters.
Morphological Processing

- Morphological operations are originally developed for bilevel images for shape and structural manipulations.
- Basic functions are *dilation* and *erosion*.
- Concatenation of dilation and erosion in different orders result in more high level operations, including *closing* and *opening*.
- Morphological operations can be used for smoothing or edge detection or extraction of other features.
- Belongs to the category of *spatial domain filter*. 
Morphological Filters for Bilevel Images

- A binary image can be considered as a set by considering “black” pixels (with image value “1”) as elements in the set and “white” pixels (with value “0”) as outside the set.

- Morphological filters are essentially set operations.
Basic Set Operations

- Let $x, y, z, \ldots$ represent locations of 2D pixels, e.g. $x = (x_1, x_2)$, $S$ denote the complete set of all pixels in an image, let $A, B, \ldots$ represent subsets of $S$.
- **Union (OR)**
  \[ A \cup B = \{x : x \in A \text{ or } x \in B\} \]
- **Intersection (AND)**
  \[ A \cap B = \{x : x \in A \text{ and } x \in B\} \]
Basic Set Operations

- Complement
  \[ A^c = \{ x : x \in S \text{ and } x \notin A \} \]

- Translation
  \[ (A)_x = \{ z : z = y + x, \ y \in A \} \]

- Reflection
  \[ \hat{A} = \{ y : y = -x, \ x \in A \} \]
Dilation

- **Dilation** of set $F$ with a structuring element $H$ is represented by $F \oplus H$

  $$F \oplus H = \{x : (\hat{H})_x \cap F \neq \Phi\}$$

  where $\Phi$ represents the empty set.

- $G = F \oplus H$ is composed of all the points that when $\hat{H}$ shifts its origin to these points, at least one point of $\hat{H}$ is included in $F$.

- If the **origin** of $H$ takes value “1”,

  $$F \subset F \oplus H$$
Example of Dilation (1)

Dilation enlarges a set.

H, 3x3, origin at the center

H, 5x3, origin at the center

Dilation enlarges a set.
Example of Dilation (2)

Note that the narrow ridge is closed

F

H, 3x3, origin at the center

G
Erosion

• Erosion of set $F$ with a structuring element $H$ is represented by $F \ominus H$, and is defined as,

$$F \ominus H = \{x : (H)_x \subseteq F\}$$

• $G = F \ominus H$ is composed of points that when $H$ is translated to these points, every point of $H$ is contained in $F$.

• If the origin of $H$ takes value of “1”, $F \ominus H \subseteq F$
Example of Erosion (1)

H, 3x3, origin at the center

H, 5x3, origin at the center

Erosion shrinks a set
Example of Erosion (2)

H, 3x3, origin at the center
Structuring element

- The shape, size, and orientation of the structuring element depend on application.
- A symmetrical one will enlarge or shrink the original set in all directions.
- A vertical one, will only expand or shrink the original set in the vertical direction.
Properties of Dilation and Erosion Operators

- **Communitivity**

\[ A \oplus B = B \oplus A, \quad \text{but} \quad A \ominus B \neq B \ominus A. \]

\[ A \oplus B = \{ x : (\hat{B})_x \cap A \neq \Phi \} = \{ x : \exists y, y \in A, y \in (\hat{B})_x \} = \{ x : \exists y, y \in A, y - x \in \hat{B} \} = \{ x : \exists y, y \in A, x - y \in B \} = \{ x : \exists z, x - z \in A, z \in B \} = \{ x : \exists z, z - x \in \hat{A}, z \in B \} = \{ x : \exists z, z \in (\hat{A})_x, z \in B \} = \{ x : (\hat{A})_x \cap B \neq \Phi \} = B \ominus A \]

- **Duality**

\[ \overline{A \ominus B} = \overline{A} \ominus \overline{B}, \quad \overline{A \oplus B} = \overline{A} \ominus \overline{B}. \]
Properties of Dilation and Erosion Operators

• Distributivity

\[ A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C), \]
\[ A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C), \]
\[ (A \cap B) \ominus C = (A \ominus C) \cap (B \ominus C). \]

• Chain rule

\[ (A \oplus B) \oplus C = A \oplus (B \oplus C), \]
\[ (A \ominus B) \ominus C = A \ominus (B \ominus C). \]
Closing and Opening

- **Closing**

  \[ F \star H = (F \bigoplus H) \bigodot H \]

  - Smooth the contour of an image
  - Fill small gaps and holes

- **Opening**

  \[ F \circ H = (F \bigodot H) \bigoplus H \]

  - Smooth the contour of an image
  - Eliminate false touching, thin ridges and branches
Example of Closing

$F$, $F \oplus H$, $(F \oplus H) \Theta H$

$H$, 3x3, origin at the center
Example of Opening

\[ F \]

\[ F \Theta H \]

\[ (F \Theta H) \oplus H \]

\[ H, \text{3x3, origin at the center} \]
Example of Opening and Closing

FIGURE 9.10
Morphological opening and closing. The structuring element is the small circle shown in various positions in (b). The dark dot is the center of the structuring element.
Properties of Opening and Closing Operators

• Duality

\[ F \bullet H = \overline{F \circ \hat{H}}, \quad F \circ H = \overline{F \bullet \hat{H}}. \]

• Idempotence

\[ (F \bullet H) \bullet H = F \bullet H, \quad (F \circ H) \circ H = F \circ H. \]

• Smoothing

– Removal of small holes and narrow branches can be accomplished by concatenating opening with closing: \[ G = (F \circ H) \bullet H \]
Morphological Filters for Grayscale Images

- The structure element $h$ is a 2D grayscale image with a finite domain $(D_h)$, similar to a filter.
- The morphological operations can be defined for both continuous and discrete images.
Dilation for Grayscale Image

- Dilation

\[(f \oplus h)(x, y) = \max \{f(x-s, y-t) + h(s, t); (s, t) \in D_h, (x-s, y-t) \in D_f\}\]

- Similar to linear convolution, with the max operation replacing the sums of convolution and the addition replacing the products of convolution.
- The dilation chooses the maximum value of \(f+h\) in a neighborhood of \(f\) defined by the domain of \(h\).
- If all values of \(h\) are positive, then the output image tends to be brighter than the input, dark details (e.g. dark dots/lines in a white background) are either reduced or eliminated.
Illustration of 1-D Grayscale Dilation

\[(f \oplus b)(s) = \max \{f(s-x) + b(x); x \in D_b\} = \max \{f(x) + b(s-x); s-x \in D_b\}\]

\[D_b = [-r; r]\]
\[s-r \leq x \leq s+r\]

**Wrong result!**

**FIGURE 9.27** (a) A simple function. (b) Structuring element of height A. (c) Result of dilation for various positions of sliding b past f. (d) Complete result of dilation (shown solid).
• Show correct result  (in Red!)

**FIGURE 9.27**  (a) A simple function.  (b) Structuring element of height \( A \).  (c) Result of dilation for various positions of sliding \( b \) past \( f \).  (d) Complete result of dilation (shown solid).
Erosion for Grayscale Image

- Erosion

\[(f \ominus h)(x, y) = \min\{f(x + s, y + t) - h(s, t); (s, t) \in D_h, (x + s, y + t) \in D_f\}\]

- Similar to linear correlation, with the \(\min\) operation replacing the sums of correlation and the subtraction replacing the products of correlation.
- The erosion chooses the minimum value of \(f-h\) in a neighborhood of \(f\) defined by the domain of \(h\).
- If all values of \(h\) are positive, then the output image tends to be darker than the input, brighter details (e.g. white dots/lines in a dark background) are either reduced or eliminated.

Yao Wang, NYU-Poly
Illustration of 1-D Grayscale Erosion

\[(f \Theta h)(s) = \min\{f(s + x) - b(x); x \in D_b\}\]

**Figure 9.28**
Erosion of the function shown in Fig. 9.27(a) by the structuring element shown in Fig. 9.27(b).

Wrong result!
• **Show correct result**

**FIGURE 9.28**
Erosion of the function shown in Fig. 9.27(a) by the structuring element shown in Fig. 9.27(b).
Example of Grayscale Dilation and Erosion

FIGURE 9.29
(a) Original image. (b) Result of dilation. (c) Result of erosion. (Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)
Opening for Grayscale Image

• Opening

\[ f \circ h = (f \ominus h) \oplus h \]

• Geometric interpretation
  – Think \( f \) as a surface where the height of each point is determined by its gray level.
  – Think the structure element has a gray scale distribution as a half sphere.
  – Opening is the surface formed by the highest points reached by the sphere as it rolls over the entire surface of \( f \) from underneath.
Closing for Grayscale Image

• Closing

\[ f \bullet h = (f \oplus h) \ominus h \]

• Geometric interpretation
  
  – Think \( f \) as a surface where the height of each point is determined by its gray level.
  
  – Think the structure element has a gray scale distribution as a half sphere.
  
  – Closing is the surface formed by the lowest points reached by the sphere as it slides over the surface of \( f \) from above.
Illustration of 1-D Grayscale Opening and Closing

**FIGURE 9.30**
(a) A gray-scale scan line.
(b) Positions of rolling ball for opening.
(c) Result of opening.
(d) Positions of rolling ball for closing.
(e) Result of closing.

**Opening**
Eliminate false touching, thin ridges and branches

**Closing**
Fill small gaps and holes
Example of Grayscale Opening and Closing

**FIGURE 9.31** (a) Opening and (b) closing of Fig. 9.29(a). (Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)
Morphological Operation for Image Enhancement

- Morphological smoothing
  - Opening followed by closing,
  - Attenuated both bright and dark details

![Diagram of morphological operation and examples](image)

*Figure 9.11*
(a) Noisy image.
(b) Eroded image.
(c) Opening of $A$.
(d) Dilation of the opening.
(e) Closing of the opening (Original image for this example courtesy of the National Institute of Standards and Technology.)
Morphological Operation for Image Enhancement

• Morphological gradient \((f \oplus h) - (f \ominus h)\)  
  – The difference between the dilated and eroded images,

• Valley detection \(f \cdot h - f\)  
  – Detect dark text/lines from a gray background

• Boundary detection \(f - f \ominus h\)
Application in Face Detection

- Use color information to detect candidate face region
- Verify the existence of face in candidate region

Input image  Skin-tone color likelihood  Opening processed image  Blob coloring  Face detection result
Written Assignment

1. For the image A in Figure 1(b), using the structuring element B in Figure 1(a), determine the closing and opening of A by B.

![Figure 1. (a)](image)

![Figure 1. (b)](image)

2. Consider a contiguous image function \( f \) and a gray scale structuring element \( h \) described by

\[
f(x, y) = \begin{cases} 
1 & -4 \leq x, y \leq 4; \\
0 & otherwise
\end{cases} \quad \text{and} \quad h(x, y) = \begin{cases} 
1 & -1 \leq x, y \leq 1; \\
0 & otherwise
\end{cases}
\]

Derive the gray scale dilation and erosion of \( f \) by \( h \), respectively.
Computer Assignment

1. Write a program which can i) add salt-and-pepper noise to an image with a specified noise density,  ii) perform median filtering with a specified window size. Consider only median filter with a square shape. Try two different noise density (0.05, 0.2) and for each density, comment on the effect of median filtering with different window sizes and experimentally determine the best window size. You can use imnoise( ) to generate noise. You should write your own function for median filtering. You can ignore the boundary problem by only performing the filtering for the pixels inside the boundary.

2. In a previous assignment you have created a program for adding Gaussian noise and filtering using average filter. Apply the averaging filter and the median filter both to an image with Gaussian noise (with a chosen noise variance) and with salt-and-pepper noise (with a chosen noise density). Comment on the effectiveness of each filter on each type of noise.

3. Use the MATLAB program imdilate( ) and imerode( ) on a sample BW image. Try a simple 3x3 square structuring element. Comment on the effect of dilation and erosion. By concatenating the dilation and erosion operations, also generate result of closing and opening, and comment on their effects. Repeat above with a 7x7 structure element. Comment on the effect of the window size. You can generate a binary image from a grayscale one by thresholding.

4. Repeat above on a gray scale image, using MATLAB gray scale dilation and erosion functions.

5. Optional (for your own practice, will not be graded): write your own program for gray-scale dilation and erosion.
Reading