EL512 --- Image Processing

Geometric Transformations: Warping, Registration, Morphing

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With contribution from Zhu Liu, Onur Guleryuz, and
Partly based on
A. K. Jain, Fundamentals of Digital Image Processing

Lecture Outline

- Introduction
- Image deformation model
- Image warping
- Image registration
- Image morphing

What is Geometric Transformation?

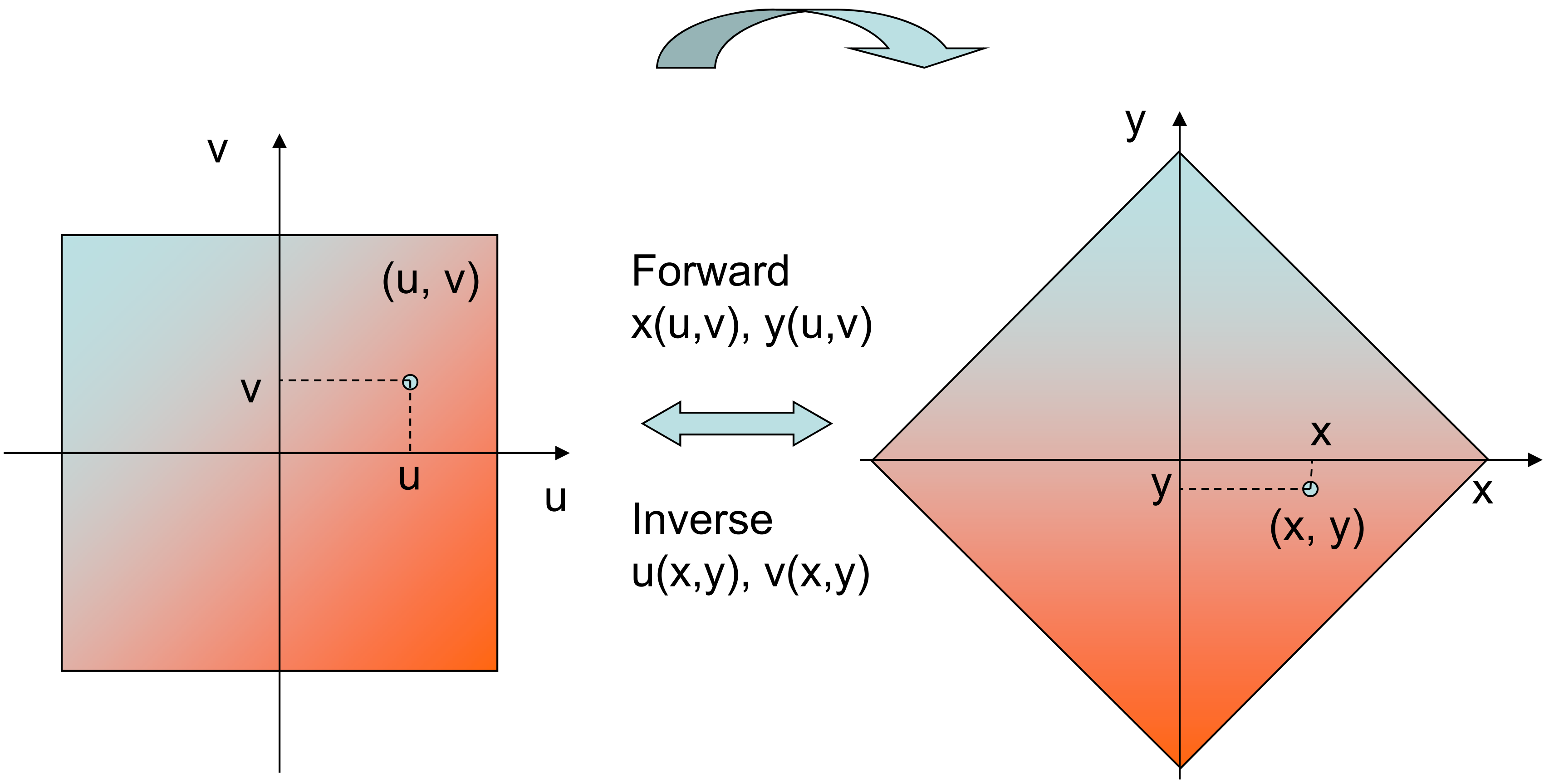
- So far, the image processing operations we have discussed modify the color values of pixels in a given image
- With geometric transformation, we modify the positions of pixels in a image, but keep their colors unchanged
  - To create special effects
  - To register two images taken of the same scene at different times
  - To morph one image to another

How to define a geometric transformation?

- Let (u, v) represent the image coordinate in an original image, and (x, y) in a deformed (or warped) image. We use a function pair to relate corresponding pixels in the two images:
  - Forward mapping: $$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}, \quad or \quad \mathbf{x} = x(\mathbf{u})$$
  - Inverse mapping: $$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}, \quad or \quad \mathbf{u} = u(\mathbf{x})$$
- Let f(u, v) or f(**u**) denote the original image and g(x, y) or g(**x**) the deformed image. Then they are related by:

$$\begin{cases} g(x,y) = f(u(x,y), v(x,y)) \\ f(u,v) = g(x(u,v), y(u,v)) \end{cases}, \quad or \quad \begin{cases} g(\mathbf{x}) = f(u(\mathbf{x})) \\ f(\mathbf{u}) = g(x(\mathbf{u})) \end{cases}$$

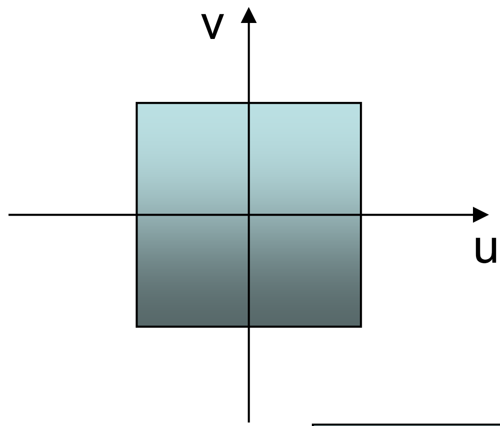
Illustration of Forward and Inverse Mapping Functions



Translation

- Translation is defined by the following mapping functions:

$$\begin{cases} x = u + t_x \\ y = v + t_y \end{cases} \quad and \quad \begin{matrix} u = x - t_x \\ v = y - t_y \end{matrix}$$

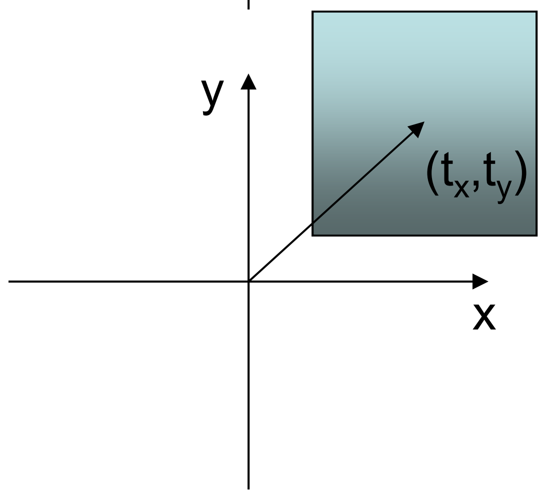


- In matrix notation

$$\mathbf{x} = \mathbf{u} + \mathbf{t}, \quad \mathbf{u} = \mathbf{x} - \mathbf{t}$$

where

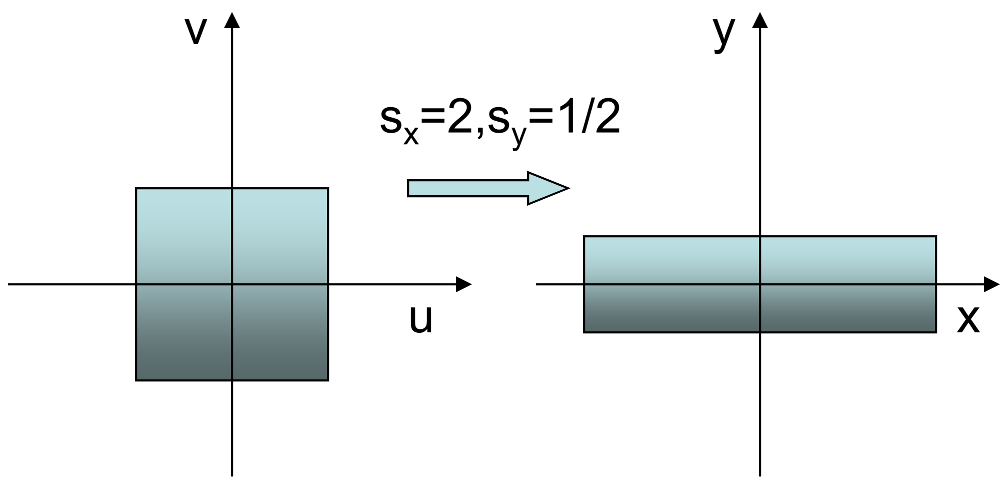
$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}.$$



Scaling

- Scaling is defined by

$$\begin{cases} x = s_x u \\ y = s_y v \end{cases} \quad and \quad \begin{cases} u = x / s_x \\ v = y / s_y \end{cases}$$



- Matrix notation

$$\mathbf{x} = \mathbf{S}\mathbf{u}, \quad \mathbf{u} = \mathbf{S}^{-1}\mathbf{x}$$

where

$$\mathbf{S} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

- If $s_x < 1$ and $s_y < 1$, this represents a minification or shrinking, if $s_x > 1$ and $s_y > 1$, it represents a magnification or zoom.

Rotation

- Rotation by an angle of θ is defined by

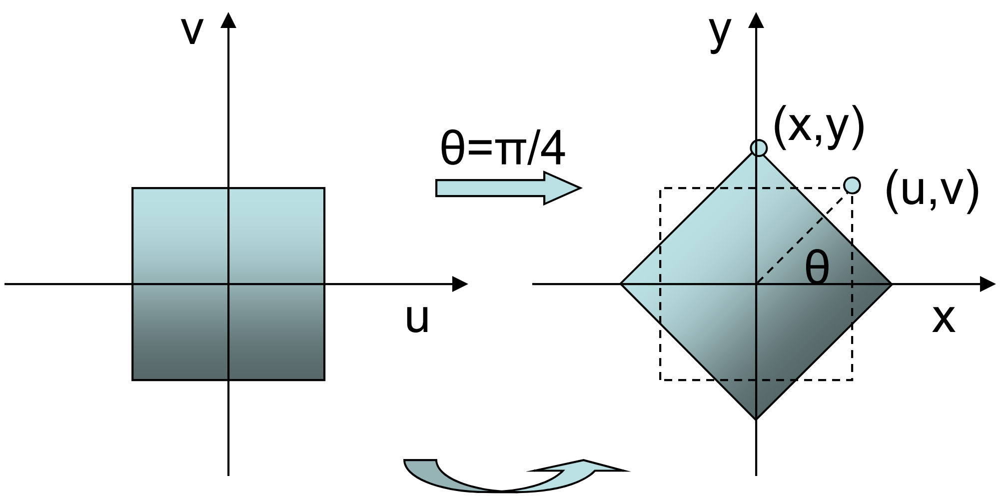
$$\begin{cases} x = u\cos\theta - v\sin\theta \\ y = u\sin\theta + v\cos\theta \end{cases} \quad and \quad \begin{cases} u = x\cos\theta + y\sin\theta \\ v = -x\sin\theta + y\cos\theta \end{cases}$$

- In matrix format

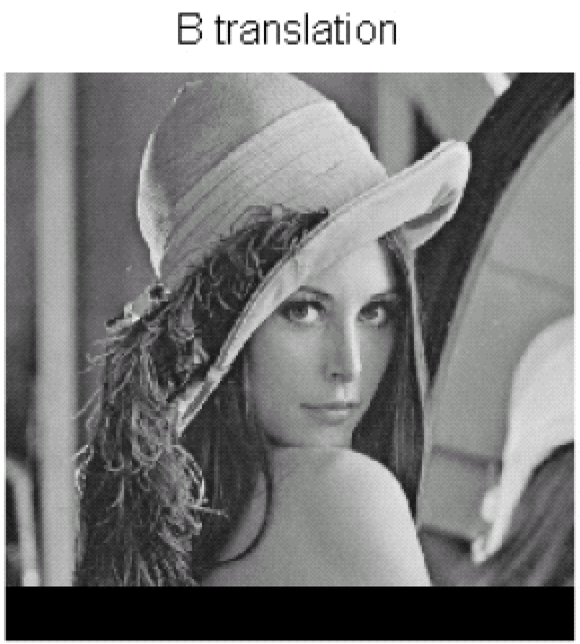
$$\mathbf{x} = \mathbf{R}\mathbf{u}, \quad \mathbf{u} = \mathbf{R}^T\mathbf{x}$$

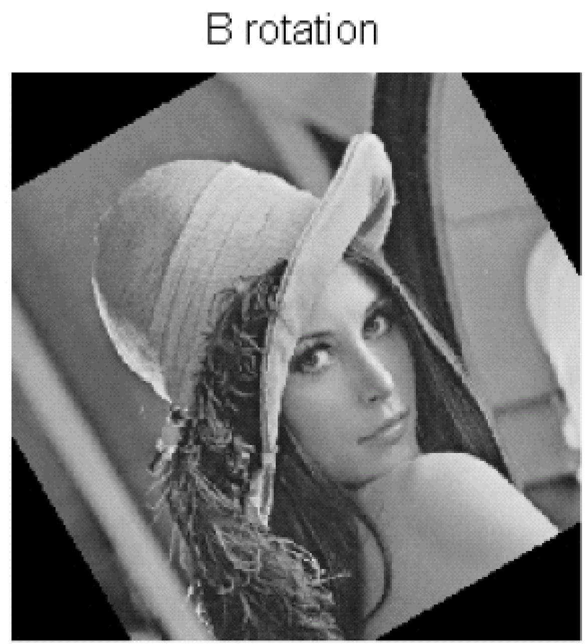
where

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



- **R** is a unitary matrix: $\mathbf{R}^{-1}=\mathbf{R}^{T}$





Translation: $x(k,l) = k + 50; y(k,l) = l;$

Rotation: $x(k,l) = (k - x_0)cos(\theta) + (l - y_0)sin(\theta) + x_0;$

$y(k,l) = -(k - x_0)sin(\theta) + (l - y_0)cos(\theta) + y_0;$

$x_0 = y_0 = 256.5$ the center of the image $\mathbf{A}$, $\theta = \pi/6$

By Onur Guleyuz

Geometric Transformation

- A geometric transformation refers to a combination of translation, scaling, and rotation, with a general form of

$$\mathbf{x} = \mathbf{RS}(\mathbf{u} + \mathbf{t}) = \mathbf{Au} + \mathbf{b},$$
$$\mathbf{u} = \mathbf{A}^{-1}(\mathbf{x} - \mathbf{b}) = \mathbf{A}^{-1}\mathbf{x} + \mathbf{c},$$
$$\textit{with } \mathbf{A} = \mathbf{RS}, \mathbf{b} = \mathbf{RSt}, \mathbf{c} = -\mathbf{t}.$$

- Note that interchanging the order of operations will lead to different results.

Affine Mapping

- All possible geometric transformations are special cases of the *Affine Mapping*:

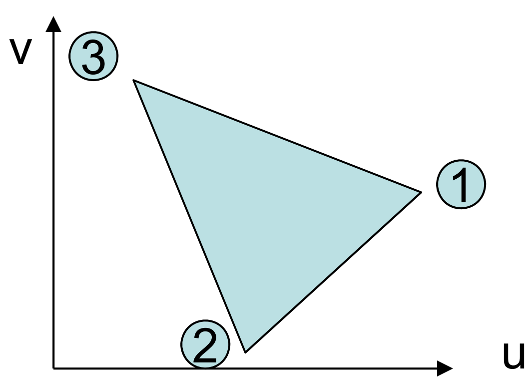
$$\begin{cases} x = a_0 + a_1 u + a_2 v \\ y = b_0 + b_1 u + b_2 v \end{cases} \quad or \quad \mathbf{x} = \mathbf{A}\mathbf{u} + \mathbf{b}$$

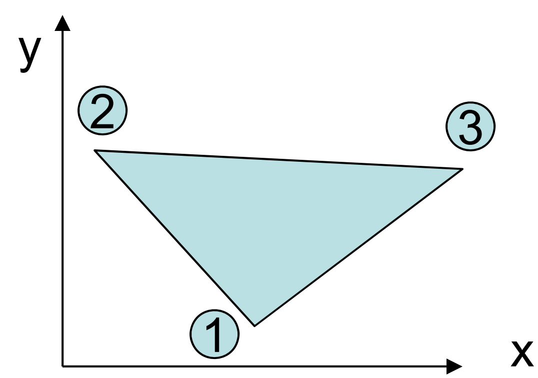
$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

- When A is a orthonormal matrix, it corresponds to a rotation matrix, and the corresponding affine mapping reduces to a geometric mapping.

Affine Mapping Properties

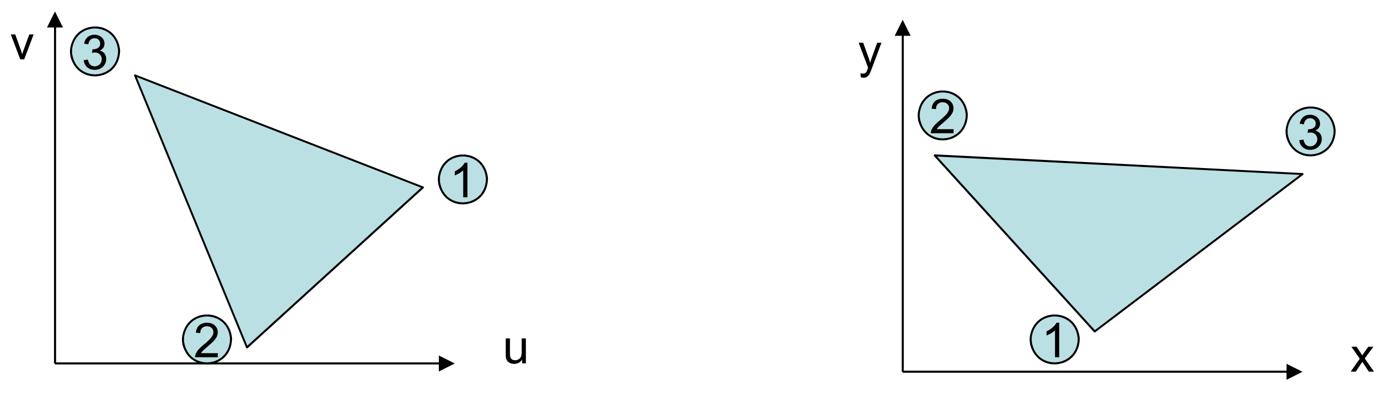
- Mapping straight lines to straight lines
- The mapping between two arbitrary triangles can be captured by an affine mapping
  - Affine mapping coefficients can be uniquely determined from displacements of 3 vertices





How to Determine Affine Parameters from Vertex Correspondences ?

- Go through in class



Matlab Functions

- T = MAKETFORM('affine',U,X) builds a TFORM struct for a
- two-dimensional affine transformation that maps each row of U
- to the corresponding row of X. U and X are each 3-by-2 and
- define the corners of input and output triangles. The corners
- may not be collinear.
- Example
- -------
- Create an affine transformation that maps the triangle with vertices
- (0,0), (6,3), (-2,5) to the triangle with vertices (-1,-1), (0,-10),
- (4,4):
-
- u = [0   6 -2]';
- v = [0   3   5]';
- x = [-1   0   4]';
- y = [-1 -10   4]';
- tform = maketform('affine',[u v],[x y]);

- G = MAKETFORM('affine',T) builds a TFORM struct G for an N-dimensional affine transformation. T defines a forward transformation such that TFORMFWD(U,T), where U is a 1-by-N vector, returns a 1-by-N vector X such that X = U * T(1:N,1:N) + T(N+1,1:N).T has both forward and inverse transformations. N=2 for 2D image transformation

In MATLAB notation

$$T = \begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_0 & b_0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}^T & 0 \\ \mathbf{b}^T & 1 \end{bmatrix}$$

Geometric Transform EL512 Image Processing 15

- B = IMTRANSFORM(A,TFORM, INTERP) transforms the image A according to the 2-D spatial transformation defined by TFORMB; INTERP specifies the interpolation filter
- Example 1
- ---------
- Apply a horizontal shear to an intensity image.
-
- I = imread('cameraman.tif');
- tform = maketform('affine',[1 0 0; .5 1 0; 0 0 1]);
- J = imtransform(I,tform);
- figure, imshow(I), figure, imshow(J)

- Show in class

Horizontal Shear Example

tform = maketform('affine',[1 0 0; .5 1 0; 0 0 1]);
In MATLAB, 'affine' transform is defined by:
[a1,b1,0;a2,b2,0;a0,b0,1]

With notation used in this lecture note

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

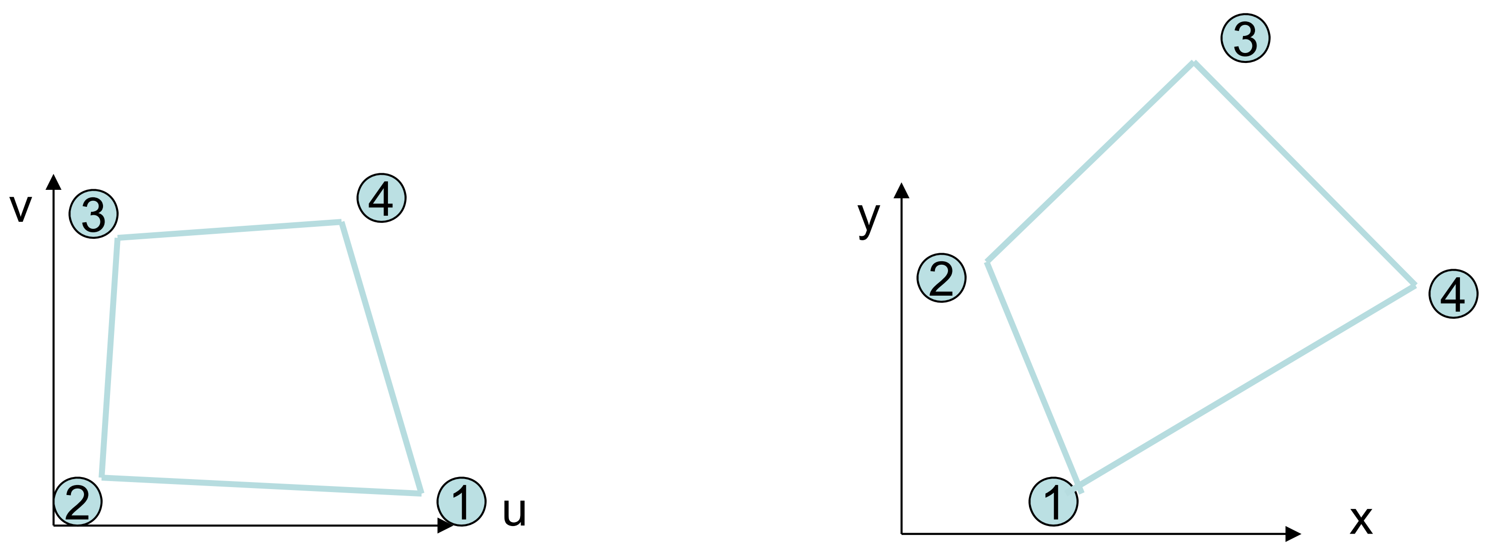
Note in this example, first coordinate indicates horizontal position, second coordinate indicate vertic

Bilinear Mapping

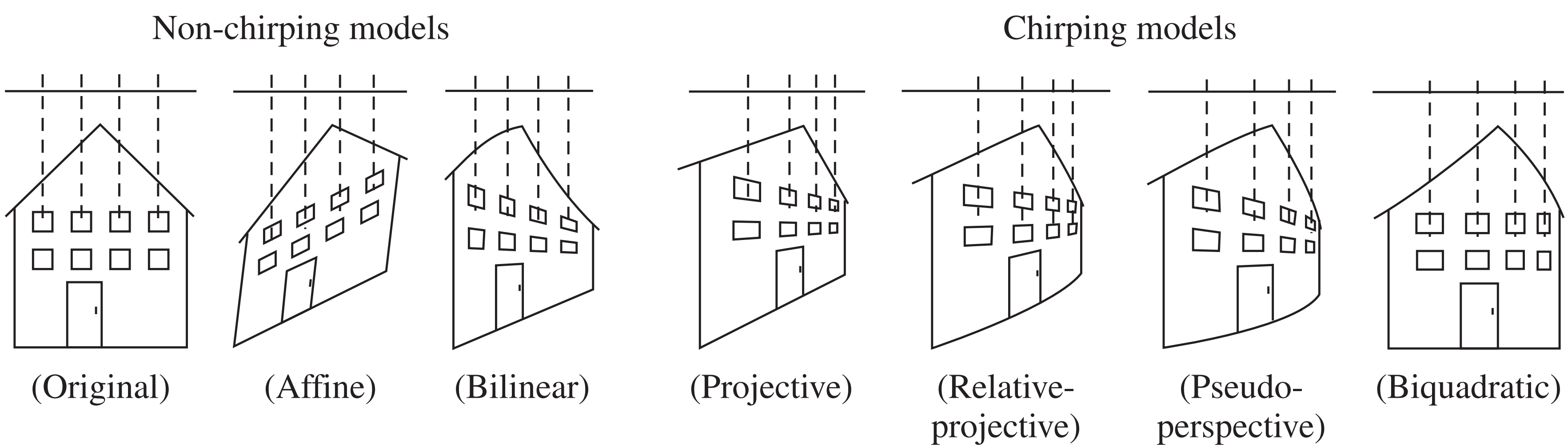
$$\begin{cases} x = a_0 + a_1 u + a_2 v + a_3 uv \\ y = b_0 + b_1 u + b_2 v + b_3 uv \end{cases}$$

- Mapping a square to a quadrangle.
- More generally mapping a quadrangle to another quadrangle.
- 8 parameters.
- Can be completely determined from how the four corners moved.

How to determine the bilinear coefficients from the vertex correspondences?



Different Types of Mapping



Two features of projective mapping:

- Chirping: increasing perceived spatial frequency for far away objects
- Converging (Keystone): parallel lines converge in distance

Polynomial Warping

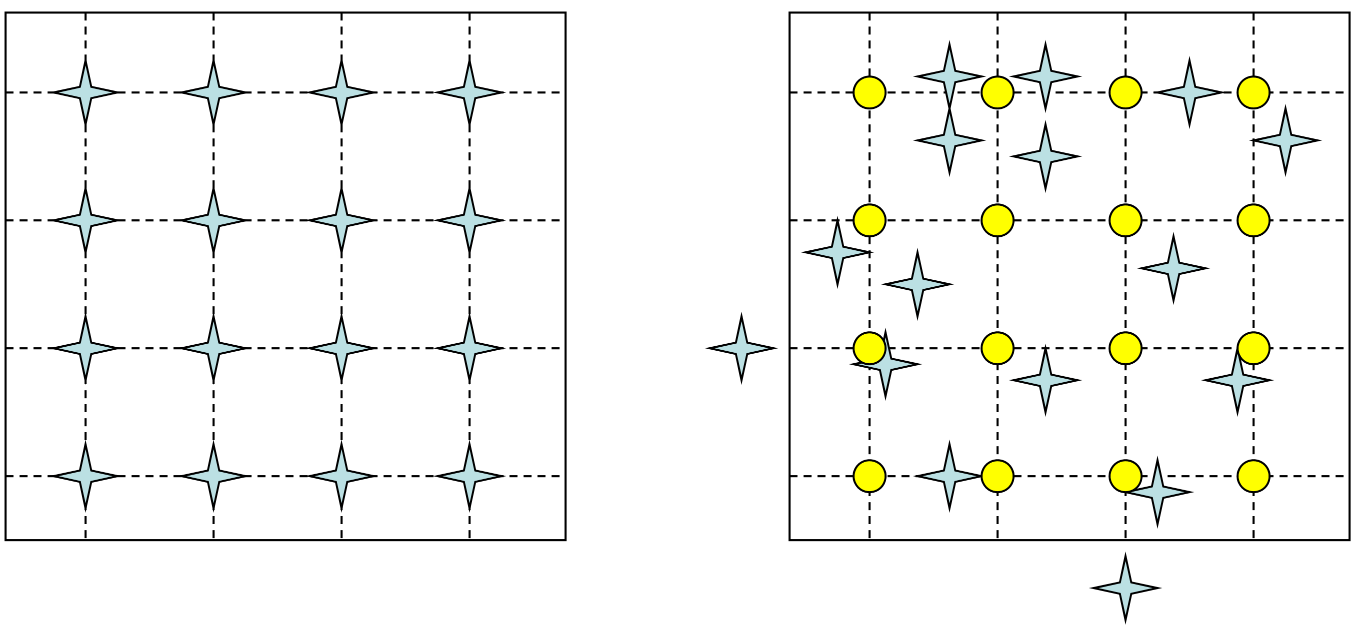
- The polynomial warping includes all deformations that can be modeled by polynomial transformations:

$$\begin{cases} x = a_0 + a_1 u + a_2 v + a_3 uv + a_4 u^2 + a_5 v^2 + \cdots \\ y = b_0 + b_1 u + b_2 v + b_3 uv + b_4 u^2 + b_5 v^2 + \cdots \end{cases}$$

- Includes affine and bilinear mapping as special cases

Image Warping by Forward Mapping

- Mapping image f(u, v) to g(x, y) based on a given mapping function: x(u, v), y(u, v).
- Forward Mapping
  - For each point (u, v) in the original image, find the corresponding position (x, y) in the deformed image by the forward mapping function, and let g(x,y)=f(u,v).
  - What if the mapped position (x,y) is not an integer sample in the desired image?

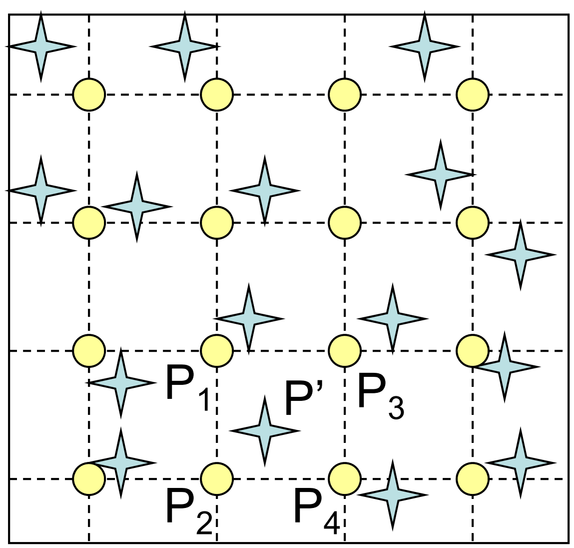


Warping points are often non-integer samples

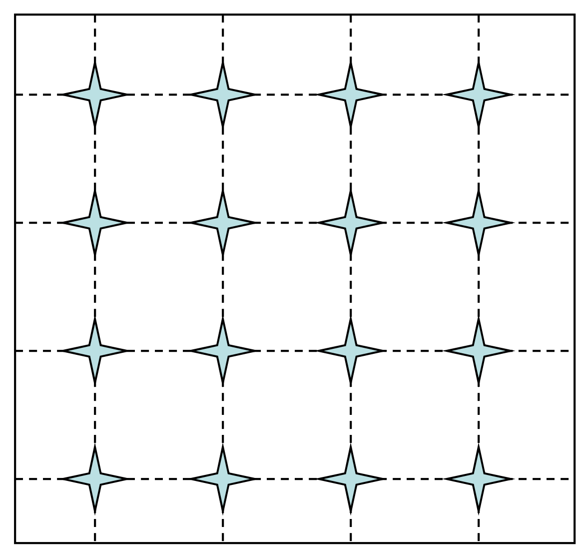
Many integer samples “o” are not assigned Values

Image Warping by Inverse Mapping

- For each point (x, y) in the image to be obtained, find its corresponding point (u, v) in the original image using the inverse mapping function, and let g(x, y) = f(u, v).
- What if the mapped point (u,v) is not an integer sample?
  – Interpolate from nearby integer samples!



P' will be interpolated from P_1, P_2, P_3, and P_4



Interpolation Method

- Nearest neighbor:
  - Round (u,v) to the nearest integer samples
- Bilinear interpolation:
  - find four integer samples nearest to (u,v), apply bilinear interpolation
- Other higher order interpolation methods can also be used
  - Requiring more than 4 nearest integer samples!

How to find inverse mapping

- Invert the forward mapping

$$\text{If} \quad \mathbf{x} = \mathbf{A}\mathbf{u} + \mathbf{b}$$
$$\text{Then} \ \mathbf{u} = \mathbf{A}^{-1}(\mathbf{x} - \mathbf{b})$$

- Directly finding inverse mapping from point correspondence

MATLAB function: interp2

- ZI = INTERP2(X,Y,Z,XI,YI, METHOD) interpolates to find ZI, the values of the underlying 2-D function Z at the points in matrices XI and YI.
  - Matrices X and Y specify the points at which the data Z is given.
  - METHOD specifies interpolation filter
    - 'nearest' - nearest neighbor interpolation
    - 'linear' - bilinear interpolation
    - 'spline' - spline interpolation
    - 'cubic' - bicubic interpolation as long as the data is uniformly spaced, otherwise the same as 'spline'

Using ‘interp2’ to realize image warping

- Use inverse mapping
- Step 1: For all possible pixels in output image (x,y), find corresponding points in the input image (u,v)
    - (X,Y)=meshgrid(1:M,1:N)
    - Apply inverse mapping function to find corresponding (u,v), for every (x,y), store in (UI,VI)
        - Can use tforminv( ) function if you derived the transformation using maketform().
        - Or write your own code using the specified mapping
- Step 2: Use interp2 to interpret the value of the input image at (UI,VI) from their values at regularly sampled points (U,V)
    - (U,V)=meshgrid(1:M,1:N)
    - Outimg=interp2(U,V,inimg,UI,VI,’linear’);

MATLAB function for image warping

- B = IMTRANSFORM(A,TFORM, INTERP) transforms the image A according to the 2-D spatial transformation defined by TFORM
- INTERP specifies the interpolation filter
- Example 1
-     ---------
-     Apply a horizontal shear to an intensity image.
-
-         I = imread('cameraman.tif');
-         tform = maketform('affine',[1 0 0; .5 1 0; 0 0 1]);
-         J = imtransform(I,tform);
-         figure, imshow(I), figure, imshow(J)

Horizontal Shear Example

tform = maketform('affine',[1 0 0; .5 1 0; 0 0 1]);
In MATLAB, 'affine' transform is defined by:
[a1,b1,0;a2,b2,0;a0,b0,1]

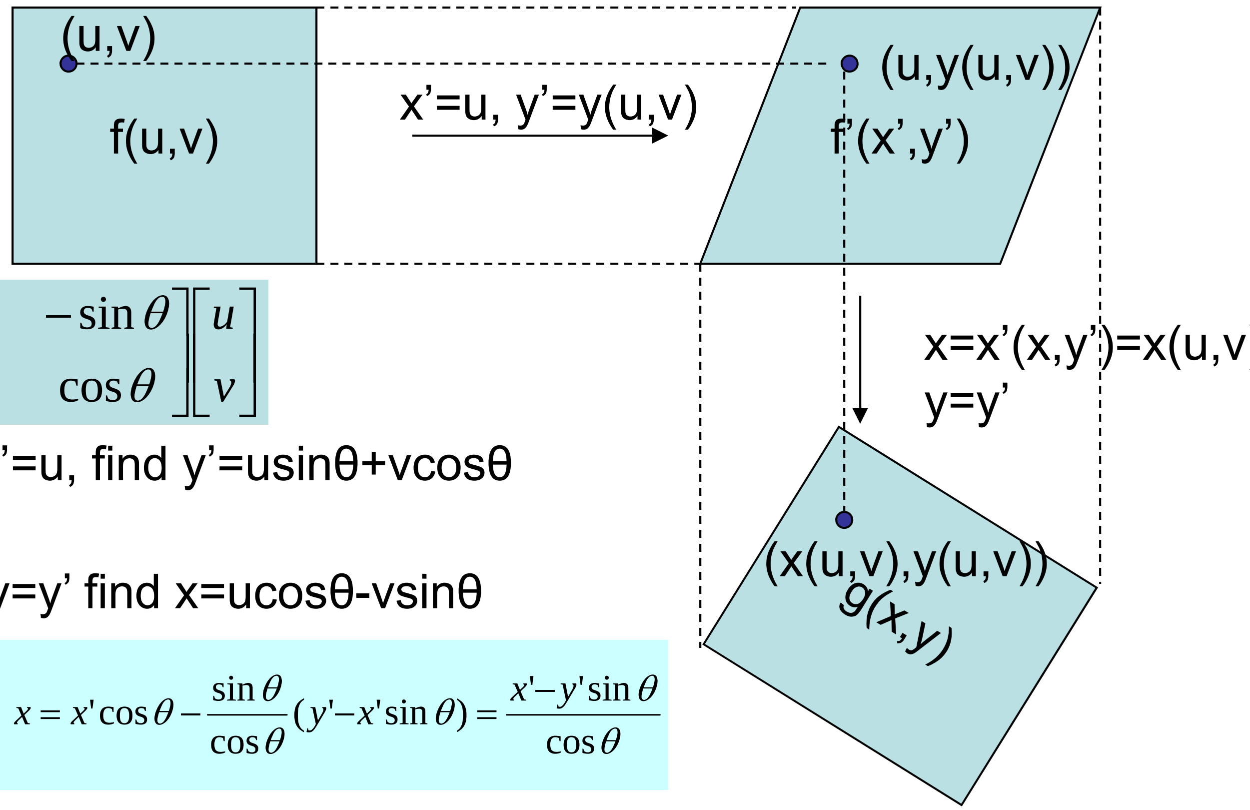
With notation used in this lecture note

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note in this example, x, u indicates vertical position, y, v indicate horizontal position

Two-Pass Mapping

- The idea is to transform each row of f(u, v) first, to obtain an intermediate image, f'(x',y'), and then each column of this intermediate image is transformed to obtain the final image g(x, y).



Ex: rotation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

Step 1: Fixed each row x'=u, find y'=usinθ+vcosθ

f'(x', y')=f(u,v)

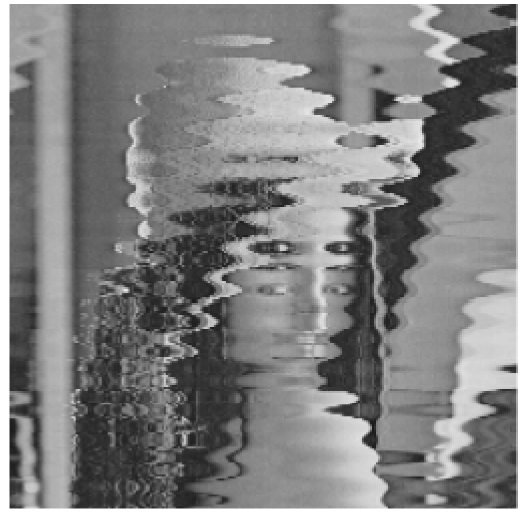
Step2: Fix each column y=y' find x=ucosθ-vsinθ

$$\begin{cases} v = \dfrac{y' - u\sin\theta}{\cos\theta} = \dfrac{y' - x'\sin\theta}{\cos\theta} \\ u = x' \end{cases} \Rightarrow \quad x = x'\cos\theta - \frac{\sin\theta}{\cos\theta}(y' - x'\sin\theta) = \frac{x' - y'\sin\theta}{\cos\theta}$$

Example of Image Warping (1)

WAVE1

WAVE2



wave1: $x(u,v)=u+20\sin(2\pi v/128); y(u,v)=v;$
wave2: $x(u,v)=u+20\sin(2\pi u/30); y(u,v)=v.$

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Example of Image Warping (2)

WARP

SWIRL

WARP

$$x(u,v) = sign(u - x_0) * (u - x_0)^2 / x_0 + x_0; y(u,v) = v$$

SWIRL

$$x(u,v) = (u - x_0)\cos(\theta) + (v - y_0)\sin(\theta) + x_0;$$

$$y(u,v) = -(u - x_0)\sin(\theta) + (v - y_0)\cos(\theta) + y_0;$$

$$r = ((u - x_0)^2 + (v - y_0)^2)^{1/2}, \theta = \pi r / 512.$$

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Image Registration

- Suppose we are given two images taken at different times of the same object. To observe the changes between these two images, we need to make sure that they are aligned properly. To obtain this goal, we need to find the correct mapping function between the two. The determination of the mapping functions between two images is known as the registration problem.
- Once the mapping function is determined, the alignment step can be accomplished using the warping methods.

How to find the mapping function?

- Assume the mapping function is a polynomial of order N
- Step 1: Identify K≥N corresponding points between two images, i.e.

$$(u_i, v_i) \leftrightarrow (x_i, y_i), i = 1,2..., K.$$

- Step 2: Determine the coefficients a_i, b_i, i = 0,…,N-1 by solving

$$\begin{cases} x(u_i, v_i) = a_0 + a_1 u_i + a_2 v_i + \cdots = x_i, \\ y(u_i, v_i) = b_0 + b_1 u_i + b_2 v_i + \cdots = y_i, \end{cases} \quad i = 1,2,..., K$$

- How to solve this?

How to Solve the Previous Equations?

- Convert to matrix equation:

$$\mathbf{Aa} = \mathbf{x}, \quad \mathbf{Ab} = \mathbf{y}$$

$$where$$

$$\mathbf{A} = \begin{bmatrix} 1 & u_1 & v_1 & \cdots \\ 1 & u_2 & v_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 1 & u_K & v_K & \cdots \end{bmatrix}, \mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{N-1} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_K \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_K \end{bmatrix}$$

If K = N, and the matrix **A** is non-singular, then

$$\mathbf{a} = \mathbf{A}^{-1}\mathbf{x}, \quad \mathbf{b} = \mathbf{A}^{-1}\mathbf{y}$$

If K > N, then we can use a least square solution

$$\mathbf{a} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{x}, \quad b = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T y$$

If K < N, or **A** is singular, then more corresponding feature points must be identified.

Examples

- If we want to use an affine mapping to register to images, we need to find 3 or more pairs of corresponding points
- If we have only 3 pairs, we can solve the mapping parameters exactly as before
- If we have more than 3 pairs, these pairs may not all be related by an affine mapping. We find the “least squares fit” by solving an over-determined system of equations

Example

MATLAB function: cp2tform()

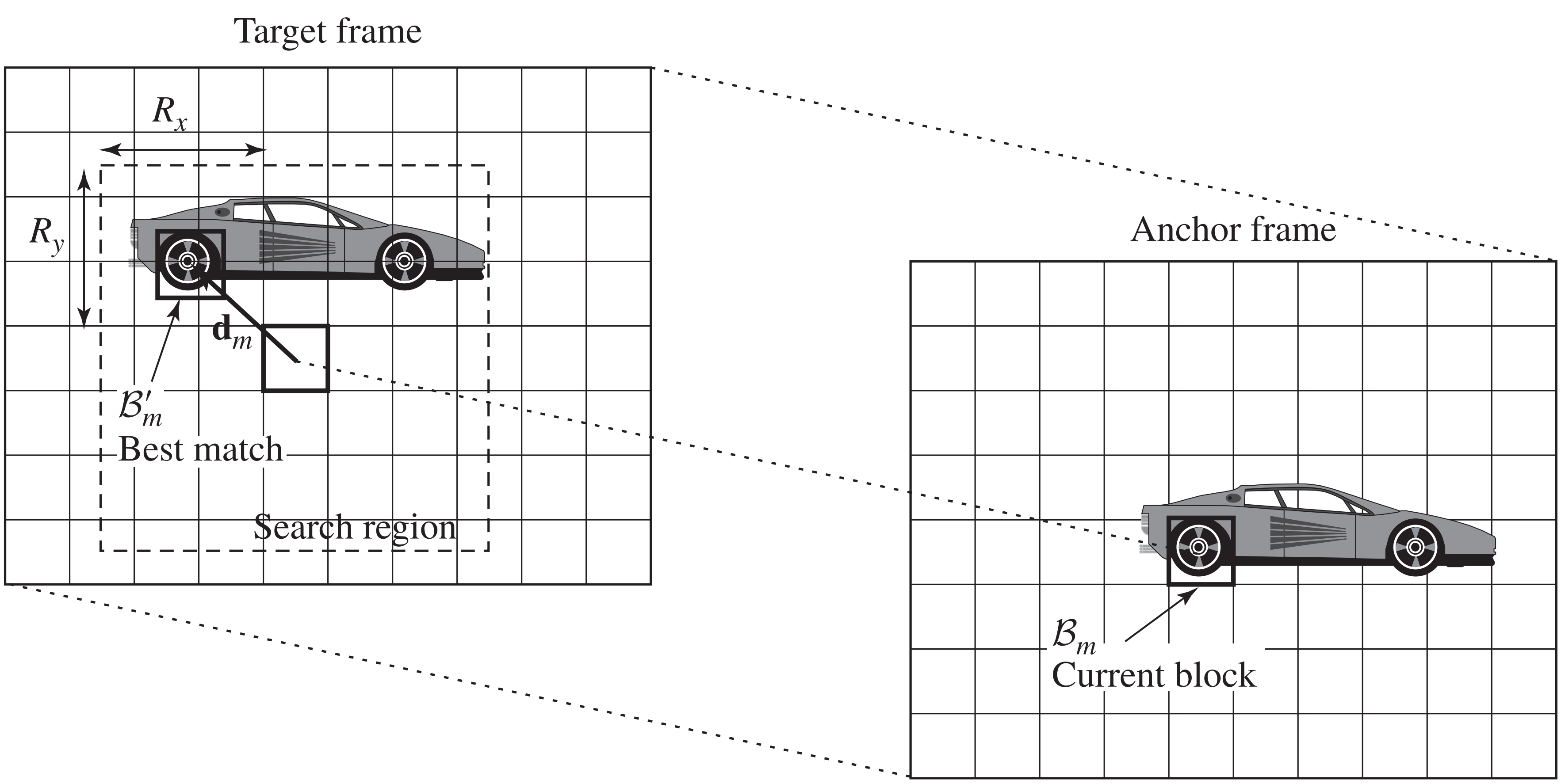
TFORM=CP2TFORM(INPUT_POINTS,BASE_POINTS,TRANSFORM TYPE)

- returns a TFORM structure containing a spatial transformation.
- INPUT_POINTS is an M-by-2 double matrix containing the X and Y coordinates of control points in the image you want to transform.
- BASE_POINTS is an M-by-2 double matrix containing the X and Y coordinates of control points in the base image.
- TRANSFORMTYPE can be 'nonreflective similarity', 'similarity', 'affine', 'projective', 'polynomial', 'piecewise linear' or 'lwm'.

How to find corresponding points in two images?

- Which points to select in one image (image 1)?
  - Ideally choose “interesting points”: corners, special features
  - Can also use points over a regular grid
- How to find the corresponding point in the other image (image 2)?
  - Put a small block around the point in image 1
  - Find a block in image 2 that matches the block pattern the best
  - Exhaustive search within a certain range.

Exhaustive Block Matching Algorithm (EBMA)



MATLAB Example

- Register an aerial photo to an orthophoto.

```
unregistered = imread('westconcordaerial.png');
figure, imshow(unregistered)
figure, imshow('westconcordorthophoto.png')
load westconcordpoints % load some points that were already picked
t_concord = cp2tform(input_points,base_points,'projective');
info = imfinfo('westconcordorthophoto.png');
registered = imtransform(unregistered,t_concord,...
                'XData',[1 info.Width], 'YData',[1 info.Height]);
figure, imshow(registered)
```

Image Morphing

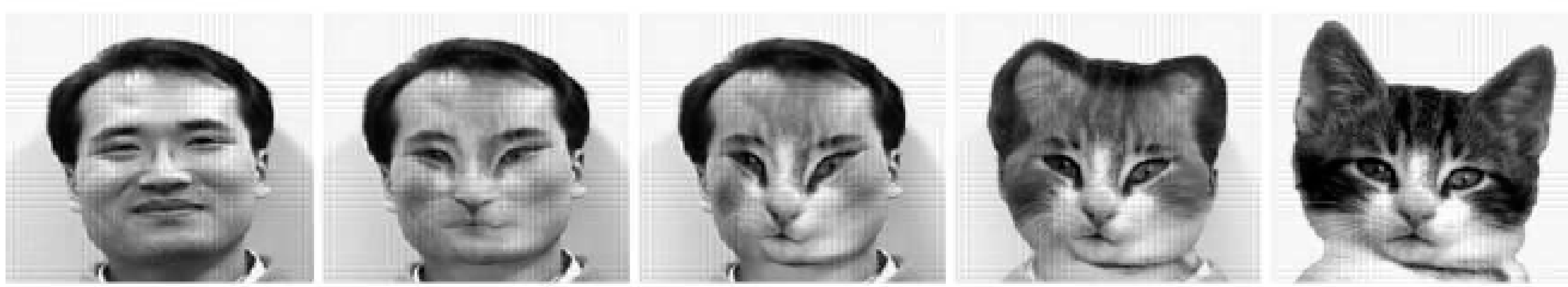
- Image morphing has been widely used in movies and commercials to create special visual effects. For example, changing a beauty gradually into a monster.
- The fundamental techniques behind image morphing is image warping.
- Let the original image be f(**u**) and the final image be g(**x**). In image warping, we create g(**x**) from f(**u**) by changing its shape. In image morphing, we use a combination of both f(**u**) and g(**x**) to create a series of intermediate images.

Examples of Image Morphing

Cross
Dissolve

$I(t) = (1-t)*S+t*T$

Mesh
based



*George Wolberg, "*Recent Advances in Image Morphing",
Computer Graphics Intl. '96, Pohang, Korea, June 1996.

Image Morphing Method

- Suppose the mapping function between the two end images is given as **x=u+d(u**). **d**(**u**) is the displacement between corresponding points in these two images.
- In image morphing, we create a series of images, starting with f(**u**) at k=0, and ending at g(**x**) at k=K. The intermediate images are a linear combination of the two end images:

$$h_k(\mathbf{u}+s_k\mathbf{d})=(1-s_k)f(\mathbf{u})+s_k g(\mathbf{u}+\mathbf{d}(\mathbf{u})),\quad k=0,1,\ldots,K,$$
$$where\ s_k=k/K.$$

MATLAB function for selecting control poins

- CPSELECT(INPUT,BASE) returns control points in CPSTRUCT. INPUT is the image that needs to be warped to bring it into the coordinate system of the BASE image.
- Example
- cpselect('westconcordaerial.png','westconcordorthophoto.png')

Demo

- Show work by senior students

Homework

- You are given two pictures of the same scene, taken at different times. In order to align the two pictures, you need to find a mapping function between the two pictures based on some common feature points. Suppose you were able to extract N (N>=3) feature points in both images that correspond to the same set of object features, with image coordinates given as (u_k,v_k) and (x_k,y_k),k=1,2,…,N. Also, suppose you want to use an affine mapping to approximate the actual unknown mapping function. How would you determine the affine mapping parameters?
- Suppose you want make a panoramic picture of a wide landscape out of two separately captured pictures with a certain overlap. Propose an algorithm for stitching up the two images to make a panorama. (list the steps involved).
- Computer assignment: Write a matlab program that implements rotation of an image by a certain angle, and apply it on a selected image. The rotation center should be the image center. Please note that you should not use the 'imrotate()' or 'imtransform()' function in MATLAB. You should write your own program, which can call "interp2()".

Reading

- Prof. Yao Wang's Lecture Notes, Chapter 10.
- R. Gonzalez, "Digital Image Processing," Section 5.11
- George Wolberg, Digital Image Warping, Wiley-IEEE Computer Society Press, 1990