Signal Representation

• What is a signal
• Time-domain description
  – Waveform representation
  – Periodic vs. non-periodic signals
• Frequency-domain description
  – Sinusoidal signals
  – Periodic signals and Fourier series representation
  – Fourier transform for non-periodic signals
  – Concepts of frequency, bandwidth, filtering
  – Numerical calculation: FFT, spectrogram
  – Demo: real sounds and their spectrogram (from DSP First)
What is a signal

- A variable (or multiple variable) that changes in time
  - Speech or audio signal: A sound amplitude that varies in time
  - Temperature variation: the temperature readings at different hours of a day
  - Stock price
  - Etc …
- More generally, a signal may vary in space and/or time
  - A picture: the color varies in 2-D space
  - A video sequence: the color varies in 2-D space and in time
- A multimedia signal consists of signals from more than one modality
  - Speech, music, image, video
- Continuous vs. Discrete time/space signal vs. digital signal
  - We will look at continuous time signal here only
Waveform Representation

• Waveform representation
  – Plot of the variable value (sound amplitude, temperature reading, stock price) vs. time
  – Mathematical representation: $s(t)$
Sample Speech Waveform

Entire waveform

```matlab
[y,fs]=wavread('morning.wav');
sound(y,fs);
figure; plot(y);
```

Blown-up of a section.

```matlab
x=y(10000:25000);plot(x);
figure; plot(x);
axis([2000,3000,-0.1,0.08]);
```

Signal within each short time interval is periodic
Periodic structure depends on the vowel being spoken
Sample Music Waveform

Entire waveform

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» [y,fs]=wavread('microsoft.wav');
» sound(y,fs);
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Blown-up of a section

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» figure; plot(y);
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```

Music has stronger periodic structure than speech
Periodic structure depends on the note being played
Sinusoidal Signals

\[ s(t) = A \cos(2\pi f_0 t + \phi) \]

- \( f_0 \) : Fundamental frequency
- \( T_0 = 1 / f_0 \) : Fundamental period
- \( A \) : Amplitude
- \( \phi \) : Phase

• Sinusoidal signals are important because they can be used to synthesize any signal
  – An arbitrary signal can be expressed as a sum of many sinusoidal signals with different frequencies, amplitudes and phases
• Pure music notes are essentially sinusoids at different frequencies
  – See more about relation between sinusoids and music in *DSP First*
Complex Exponential Signals

• Complex exponential

\[ s(t) = A \exp(j2\pi f_0 t) = |A| \cos(2\pi f_0 t + \phi) + j|A| \sin(2\pi f_0 t + \phi) \]

• Euler formula

\[ \exp(j\omega t) + \exp(-j\omega t) = 2\cos(\omega t) \]
\[ \exp(j\omega t) - \exp(-j\omega t) = j2\sin(\omega t) \]

• A complex exponential can be represented as a rotating phasor, and its horizontal/vertical projection correspond to the Real part (cos) and Imaginary part (sin).
Periodic Signals: A Square Wave

- Fundamental period \((T_0)\): The shortest interval that a signal repeats
- Fundamental frequency \((f_0)\): \(f_0 = \frac{1}{T_0}\)
Approximation by Sum of Sinusoids

2 sinusoids: 1\textsuperscript{st} and 3\textsuperscript{rd} harmonics

4 sinusoids: 1, 3, 5, 7 harmonics

View note for matlab code
Generalization to Other Periodic Signals

• Any periodic signal can be approximated by a sum of many sinusoids at harmonic frequencies of the signal \((k f_0)\) with appropriate amplitude and phase.
• The more harmonic components are added, the more accurate the approximation becomes.
• Instead of using sinusoidal signals, mathematically, we can use the complex exponential functions and take the real part.
Fourier Series Representation of Periodic Signals

Fourier Series Synthesis (inverse transform) (assuming the signal is real):

\[ s(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = S_0 + \text{Re}\left\{ \sum_{k=1}^{\infty} S_k \exp(j2\pi kf_0 t) \right\}, \]

\[ S_k = A_k e^{j\phi_k}, \quad S_0 = A_0 \]

(single sided representation)

\[ = \sum_{k=-\infty}^{\infty} T_k \exp(j2\pi kf_0 t), \quad T_k = S_k/2, \quad k > 0, \quad T_k = S_k^*/2, \quad k < 0, \quad T_0 = S_0 \]

(double sided representation)

Fourier series analysis (forward transform):

\[ S_0 = \frac{1}{T_0} \int_0^{T_0} s(t) \, dt \]

\[ S_k = \frac{2}{T_0} \int_0^{T_0} s(t) \exp(j2\pi kf_0 t) \, dt; \quad k = 1, 2, \ldots \]

or \[ T_k = \frac{1}{T_0} \int_0^{T_0} s(t) \exp(j2\pi kf_0 t) \, dt; \quad k = 0, \pm 1, 2, \ldots \]
Fourier Series Representation of Square Wave

- Applying the Fourier series analysis formula (one-sided), we get

\[ S_k = \begin{cases} 
  \frac{4}{j\pi k} & k = 1, 3, 5, \ldots \\
  0 & k = 0, 2, 4, \ldots 
\end{cases} \]

- Do the derivation on the board
The magnitude of the expansion coefficients drops exponentially, which is not very fast. The very sharp transition in square waves calls for very high frequency sinusoids to synthesize.

What is drawn on the left is the single sided spectrum. If you draw double sided spectrum, the magnitude of each line is half of the single sided spectrum, for $k=1,2,\ldots$
Approximation by Sum of Sinusoids

- 2 sinusoids: 1st and 3rd harmonics
- 4 sinusoids: 1, 3, 5, 7 harmonics

View note for matlab code
Fourier Transform for Non-Periodic Signals

A periodic signal $\rightarrow T_0 = \infty \rightarrow f_0 = 0 \rightarrow$ uncountable number of harmonics
$\rightarrow$ integral instead of sum

Fourier synthesis (forward transform):

$$s(t) = \int_{-\infty}^{\infty} S(f) \exp(j2\pi ft) df$$

Fourier analysis (inverse transform):

$$S(f) = \int_{-\infty}^{\infty} s(t) \exp(-j2\pi ft) dt$$
Pulse Function: Time Domain

\[ s(t) = \begin{cases} 
1 & -T/2 < t < T/2 \\
0 & \text{otherwise} 
\end{cases} \quad \Leftrightarrow \quad S(f) = T \frac{\sin(\pi Tf)}{\pi Tf} = T \text{sinc}(Tf) \]
Pulse Function: Spectrum

\[ s(t) = \begin{cases} 
1 & \quad -T/2 < t < T/2 \\
0 & \quad \text{otherwise}
\end{cases} \quad \iff \quad S(f) = T \frac{\sin(\pi T f)}{\pi T f} = T \ \text{sinc}(T f) \]

The peaks of the FT magnitude drops slowly. This is because the pulse function has sharp transition, which contributes to very high frequency in the signal.
Exponential Decay: Time Domain

\[ s(t) = \begin{cases} 
\exp(-\alpha t) & t > 0 \\
0 & \text{otherwise} 
\end{cases} \quad \Leftrightarrow \quad S(f) = \frac{1}{\alpha + j2\pi f}; |S(f)| = \frac{1}{\sqrt{\alpha^2 + 4\pi^2 f^2}} \]
Exponential Decay: Spectrum

\[ s(t) = \begin{cases} 
\exp(-\alpha t) & t > 0 \\
0 & \text{otherwise}
\end{cases} \iff S(f) = \frac{1}{\alpha + j2\pi f}; |S(f)| = \frac{1}{\sqrt{\alpha^2 + 4\pi^2 f^2}} \]

The FT magnitude drops much faster than for the pulse function. This is because the exponential decay function does not have a sharp transition.
Concept of (Effective) Bandwidth

- $f_{\text{min}}$ ($f_{\text{max}}$): lowest (highest) frequency where the FT magnitude is less than a threshold
- Bandwidth: $B = f_{\text{max}} - f_{\text{min}}$
- A non-bandlimited signal can be converted to bandlimited by filtering
More on Bandwidth

• Bandwidth of a signal is a critical feature when dealing with the transmission of this signal

• A communication channel usually operates only at certain frequency range (called channel bandwidth)
  – The signal will be severely attenuated if it contains frequencies outside the range of the channel bandwidth
  – To carry a signal in a channel, the signal needed to be modulated from its baseband to the channel bandwidth
  – Multiple narrowband signals may be multiplexed to use a single wideband channel
### Bandwidth of Multimedia Signals/Channels

<table>
<thead>
<tr>
<th>Signal / Channel</th>
<th>Bandwidth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Telephone speech</td>
<td>4 KHz</td>
</tr>
<tr>
<td>AM radio station</td>
<td>10 KHz</td>
</tr>
<tr>
<td>Hi-Fi amplifier</td>
<td>20 KHz</td>
</tr>
<tr>
<td>FM radio station</td>
<td>200 KHz</td>
</tr>
<tr>
<td>AM radio band</td>
<td>1.2 MHz</td>
</tr>
<tr>
<td>TV channel</td>
<td>6 MHz</td>
</tr>
<tr>
<td>FM radio band</td>
<td>20 MHz</td>
</tr>
</tbody>
</table>

From A. M. Noll, Table 4.2
How to Observe Frequency Content from Waveforms?

• A constant -> only zero frequency component (DC component)
• A sinusoid -> Contain only a single frequency component
• Periodic signals -> Contain the fundamental frequency and harmonics -> Line spectrum
• Slowly varying -> contain low frequency only
• Fast varying -> contain very high frequency
• Sharp transition -> contain from low to high frequency
• Highest frequency estimation?
  – Find the shortest interval between peak and valleys
• Go through examples on the board
Advantage of Frequency Domain Representation

- Clearly shows the frequency composition of the signal
- One can intentionally limit the bandwidth of a signal by applying a low-pass filter
  - Smooth the original signal in the temporal domain;
  - Required when sampling a fast varying signal.
- More generally, one can change the magnitude of any frequency component arbitrarily by a filtering operation
  - Lowpass -> smoothing, noise removal
  - Highpass -> edge/transition detection
  - High emphasis -> edge enhancement
- One can also shift the central frequency by modulation
  - A core technique for communication, which uses modulation to multiplex many signals into a single composite signal, to be carried over the same physical medium.
Filtering in Frequency Domain

Filtering is done by a simple multiplication:

$$G(f) = S(f) H(f)$$

$H(f)$ is designed to magnify or reduce the magnitude (and possibly phase) of the original signal at different frequencies.

A pulse signal after low pass filtering (left) will have rounded corners.
Typical Filters

- Lowpass -> smoothing, noise removal
- Highpass -> edge/transition detection
- Bandpass -> Retain only a certain frequency range
Numerical Calculation of FT

- The original signal is digitized, and then a Fast Fourier Transform (FFT) algorithm is applied, which yields samples of the FT at equally spaced intervals.
- For a signal that is very long, e.g. a speech signal or a music piece, spectrogram is used.
  - Fourier transforms over successive overlapping short intervals
Spectrogram
Sample Speech Waveform

Entire waveform

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Signal within each short time interval is periodic
Periodic structure depends on the vowel being spoken
Sample Speech Spectrogram

Signal power drops sharply at about 4KHz

Line spectra at multiple of f0, maximum frequency about 4 KHz
Sample Music Waveform

Entire waveform

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» sound(y,fs);
» figure; plot(y);

Blown-up of a section

» figure; plot(y);
» axis([1500,2500,-0.8,0.8])

Music has stronger periodic structure than speech
Periodic structure depends on the note being played
Sample Music Spectrogram

» figure; » psd(y,256,fs);

Signal power drops gradually in the entire frequency range

» figure; » specgram(y,256,fs);

Line spectra are smoother, maximum frequency above 4 KHz
Summary of Characteristics of Speech & Music

- Typical speech and music waveforms are semi-periodic
  - The fundamental period is called pitch period
  - The inverse of the pitch period is the fundamental frequency (f0)
- Spectral content
  - A speech or music signal can be decomposed into a pure sinusoidal component with frequency f0, and additional harmonic components with frequencies that are multiples of f0.
  - The maximum frequency is usually several multiples of the fundamental frequency
  - Speech has a frequency span up to 4 KHz
  - Audio has a much wider spectrum, up to 22KHz
Demo

- Demo in DSP First, Chapter 3, Sounds and Spectrograms
  - Look at the waveform and spectrogram of sample signals, while listening to the actual sound
  - Simple sounds
  - Real sounds
What Should You Know (I)

• Sinusoid signals:
  – Can determine the period, frequency, magnitude and phase of a sinusoid signal from a given formula or plot

• Fourier series for periodic signals
  – Understand the meaning of Fourier series representation
  – Can calculate the Fourier series coefficients for simple signals
  – Can sketch the (double sided) line spectrum from the Fourier series coefficients

• Fourier transform for non-periodic signals
  – Understand the meaning of the inverse Fourier transform
  – Can calculate the Fourier transform for simple signals
  – Can sketch the spectrum
  – Can determine the bandwidth of the signal from its spectrum
  – Know how to interpret a spectrogram plot
What Should You Know (II)

• Filtering concept
  – Know how to apply filtering in the frequency domain
  – Can interpret the function of a filter based on its frequency response
    • Lowpass -> smoothing, noise removal
    • Highpass -> edge detection, differentiator
    • Bandpass -> retain certain frequency band, useful for demodulation

• Speech and music signals
  – Typical bandwidth for both
  – Different patterns in the spectrogram
  – Understand the connection between music notes and sinusoidal signals
References

- McClellan, Schafer and Yoder, DSP First, Chaps. 2, 3
- Noll, Principles of Modern Communications Technology, Chap. 4.