1. For the signal $s(t) = \cos(2\pi t)$, plot its waveform, and then illustrate the resulting samples with the following sampling intervals:
   a. $T_s = 0.5$ sec.  
   b. $T_s = 1$ sec.  
   c. $T_s = 0.75$ sec.

   For each case, also sketch the reconstructed continuous time signal from the samples using linear interpolation (i.e. connecting samples by straight lines). Based on your sketch, determine the fundamental frequency of the reconstructed signal. In which case you had aliasing distortion? What is the minimal sampling frequency and the corresponding sampling interval needed to avoid aliasing?

Solution:

See figures on the next page. In each figure, the arrows indicate the samples, and solid lines are the reconstructed signals from the samples by linear interpolation. $T_r$ is the period of the reconstructed signal, and $f_r = \frac{1}{T_r}$ is the frequency of the reconstructed signal.

The signal frequency is $f_m = 1$ Hz. To avoid aliasing, the sampling frequency should be greater than the Nyquist rate $= 2f_m$, that is $f_s > 2 \times 1 = 2$ Hz. The corresponding sampling interval is $T_s = \frac{1}{f_s} < \frac{1}{2}$ sec. For part (b) and (c) the sampling interval is larger than $\frac{1}{2}$, and the reconstructed signal from samples had the frequency lower than the original (i.e. aliasing occurs). For part a), sampling interval exactly equals the $\frac{1}{2}$. In this particular example, the samples fall on the peaks and valleys of the signal and no aliasing occurs. Had the first sample started where the signal is zero, all the samples will be zeros. To avoid such problems, the sampling rate should be greater than the Nyquist rate.
\[ S(t) = \cos(2\pi t) \]

\[ a. \quad T_s = 0.5 \text{ sec.} \]

\[ \begin{align*}
T_r &= 1 \text{ sec} \\
fr &= 1 \text{ Hz}
\end{align*} \]

\[ b. \quad T_s = 1 \text{ sec.} \]

\[ \begin{align*}
T_r &= \infty \\
fr &= 0 \text{ Hz}
\end{align*} \]

\[ c. \quad T_s = 0.75 \text{ sec.} \]

\[ \begin{align*}
T_r &= 3 \text{ sec} \\
fr &= \frac{1}{3} \text{ Hz}
\end{align*} \]
2. For the sample signal and each of the sampling periods given in Prob. 1, illustrate the spectrum of the original signal and the spectrum of the sampled signal. Assuming you apply the ideal low-pass filter for reconstruction, what is the frequency of the reconstructed signal? Does your solution conform to the answers you got for Prob. 1?

Solution:

See figures on the next page for the spectra of the original signal and the sampled signals. In each spectrum plot, we draw the two lines corresponding to the original signal frequency (+/- 1 Hz), as well as lines corresponding to the aliased frequencies. The crosses ("x") indicate spectrum replication centers (at multiples of the sampling frequencies). The aliased frequencies are obtained by shifting the center of the original signal spectrum to the replication centers. The dashed lines indicate the low-pass filter for reconstruction. What remains in the pass-band of the low-pass filter are the reconstructed signal frequency. It can be seen that the results are consistent with the solution from Prob. 1 based on the temporal waveform.

You can ignore the sampling pulse train drawn on the left.
(1) $T_s = 0.5$ sec, $f_s = \frac{1}{T_s} = 2$ Hz

Sampling Impulse:

\[
\begin{array}{c}
0 \\
\vdots \\
\frac{\pi}{T_s} \\
\vdots \\
3 \\
\end{array}
\]

Sampled Signal:

\[
\begin{array}{c}
0 \\
\vdots \\
\frac{1}{T_s} \\
\vdots \\
5 \\
\end{array}
\]

Legend: * - sampling frequency

(2) $T_s = 1$ sec, $f_s = \frac{1}{T_s} = 1$ Hz

Sampling Impulse:

\[
\begin{array}{c}
-3 \\
\vdots \\
\frac{\pi}{T_s} \\
\vdots \\
3 \\
\end{array}
\]

Sampled Signal:

\[
\begin{array}{c}
-1 \\
\vdots \\
1 \\
\vdots \\
5 \\
\end{array}
\]

(3) $T_s = 0.75$ sec, $f_s = \frac{1}{T_s} = \frac{4}{3}$

Sampling Impulse:

\[
\begin{array}{c}
-4 \\
\vdots \\
\frac{\pi}{T_s} \\
\vdots \\
4 \\
\end{array}
\]

Sampled Signal:

\[
\begin{array}{c}
-3 \\
\vdots \\
1 \\
\vdots \\
5 \\
\end{array}
\]

$f_r = \frac{3}{4}$ Hz
3. For the signal below, indicate the samples obtained with a sampling interval $T=0.2$. Then sketch the reconstructed signal obtained by
a. Sample-and-hold (i.e. using rectangular pulse as interpolation kernel)
b. Linear interpolation. (i.e. using triangular pulse as interpolation kernel)

Solution
a.
4. For the sampled signal obtained in Prob. 3, first down-sample the signal by a factor of 2, without prefiltering. Next up-sample the resulting signal by a factor of 2, using sample-and-hold. Indicate on the same plot, the original samples, the samples after down-sampling, and the samples after up-sampling.

Solution:
Take every other sample for down-sampling.
Repeat the left neighbor for up-sampling.

See figures on the next page.
5. For the sampled signal obtained in Prob. 3, first down-sample the signal by a factor of 2, using the two-sample averaging filter. Next up-sample the resulting signal by a factor of 2, using linear interpolation. Indicate on the same plot, the original samples, the samples after down-sampling, and the samples after up-sampling.

Solution:
Calculate the average of current sample and its right neighbor for down-sampling. Take the average of left neighbor and right neighbor for up-sampling.

See figures on the next page.