EE3054
Signals and Systems

Continuous Time Convolution

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LECTURE OBJECTIVES

- Review of C-T LTI systems
- Evaluating convolutions
  - Examples
  - Impulses
- LTI Systems
  - Stability and causality
  - Cascade and parallel connections
Linear and Time-Invariant (LTI) Systems

- If a continuous-time system is both linear and time-invariant, then the output $y(t)$ is related to the input $x(t)$ by a convolution integral where $h(t)$ is the impulse response of the system.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = x(t) * h(t)$$
Evaluating a Convolution

\[ x(t) = u(t - 1) \]

\[ h(t) = e^{-t}u(t) \]

\[ y(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau = h(t) * x(t) \]
“Flipping and Shifting”

\[ g(\tau) = x(-\tau) = u(-\tau - 1) \]

\[ g(\tau - t) = x(-(\tau - t)) = x(t - \tau) \]
Evaluating the Integral

\[ y(t) = \begin{cases} 
0 & t - 1 < 0 \\
\int_0^{t-1} e^{-\tau} d\tau & t - 1 > 0 
\end{cases} \]
Solution

\[ y(t) = \int_{0}^{t-1} e^{-\tau} d\tau = -e^{-\tau}\bigg|_{0}^{t-1} \]
\[ = 1 - e^{-(t-1)} \quad t \geq 1 \]

\[ y(t) = 0 \quad t < 1 \]

\[ \left(1 - e^{-(t-1)}\right)u(t-1) \]
Convolution GUI

Signal Flipped Signal

Multiplication

Convolution

Input

Impulse Response

Get x(t)

Get h(t)

Flip x(t)

Flip h(t)

Signal Axis:
\[ h(\tau) = \text{blue} \]
\[ x(t-\tau) = \text{red} \]

Multiplication Axis:
\[ h(\tau)x(t-\tau) \]

Convolution Axis:
\[ y(t) = \int h(\tau)x(t-\tau) d\tau \]

Close

Help
Another Example

\[ x(t) = e^{-at}u(t) \quad h(t) = e^{-bt}u(t), \quad b \neq a \]

\[
y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t)*h(t) \\
= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau = \begin{cases} 
  e^{-bt} \int_{0}^{t} e^{-a\tau}e^{b\tau}d\tau & t > 0 \\
  0 & t < 0 
\end{cases} \\
= \begin{cases} 
  \frac{e^{-at} - e^{-bt}}{-a + b} & t > 0 \\
  0 & t < 0 
\end{cases} u(t) \\
= \frac{e^{-at} - e^{-bt}}{b - a} u(t)\]
Special Case: \( u(t) \)

\[
x(t) = e^{-at}u(t), \quad a \neq 0
\]

\[
h(t) = u(t)
\]

\[
y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) \ast h(t)
\]

\[
y(t) = \frac{1}{a} (1 - e^{-at})u(t)
\]

if \( a = 2 \)

\[
y(t) = \frac{1}{2} (1 - e^{-2t})u(t)
\]
**Convolve Unit Steps**

\[ x(t) = u(t) \quad h(t) = u(t) \]

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = x(t) * h(t) \]

\[ = \int_{-\infty}^{\infty} u(\tau) u(t-\tau) d\tau = \begin{cases} \int_0^t 1 d\tau & t > 0 \\ 0 & t < 0 \end{cases} \]

\[ = \begin{cases} t & t > 0 \\ 0 & t < 0 \end{cases} = tu(t) \]

*Unit Ramp*
“Flipping and Shifting”

\[ g(\tau) = x(-\tau) = u(-\tau - 1) \]

\[ g(\tau - t) = x(-(\tau - t)) = x(t - \tau) \]
More examples

- Rectangular pulses
Another Convolution Example

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) \, d\tau = x(t) \ast h(t) \]

\[ h(t) = e^{-t} u(t) \]
Evaluating the Integral

\[ y(t) = \begin{cases} 
0 & \text{if } t < 1 \\
\frac{t}{1} & \text{if } 1 \leq t \leq 2 \\
\frac{2}{1} & \text{if } 2 \leq t 
\end{cases} \]
Solution

\[ y(t) = \begin{cases} 
\int_{1}^{t} e^{-(t-\tau)} d\tau = e^{-(t-\tau)} \bigg|_{1}^{t} = 1 - e^{-(t-1)} & 1 \leq t \leq 2 \\
= \int_{1}^{2} e^{-(t-\tau)} d\tau = e^{-(t-\tau)} \bigg|_{1}^{2} = e^{-(t-2)} - e^{-(t-1)} & 2 \leq t 
\end{cases} \]

For \( y(t) = 0 \) when \( t < 1 \)
Convolution GUI

Signal

Flipped Signal

Multiplication

Convolution

Input

Impulse Response

Get x(t)

Get h(t)

Flip x(t)

Flip h(t)

Signal Axis:

\(x(t) = \text{blue}\)

\(h(t-t) = \text{red}\)

Multiplication Axis:

\(x(t)h(t)\)

Convolution Axis:

\(y(t) = \int x(t)h(t-t)dt\)

Close

Help
Convolution with impulses

\[ x(t) * \delta(t - t_1) = x(t - t_1) \]

Convolution with step function = integrator
Convolution is Commutative

\[ h(t) \ast x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \]

let \( \sigma = t - \tau \) and \( d\sigma = -d\tau \)

\[ h(t) \ast x(t) = -\int_{-\infty}^{\infty} h(t-\sigma) x(\sigma) d\sigma \]

\[ = \int_{-\infty}^{\infty} h(t-\sigma) x(\sigma) d\sigma = x(t) \ast h(t) \]
Stability

- A system is stable if every bounded input produces a bounded output.
- A continuous-time *LTI system* is stable if and only if

\[ \int_{-\infty}^{\infty} |h(t)| \, dt < \infty \]
Integrator is unstable
Causal Systems

- A system is causal if and only if $y(t_0)$ depends only on $x(\tau)$ for $\tau \leq t_0$.

- An LTI system is causal if and only if

$$h(t) = 0 \text{ for } t < 0$$
Convolution is Linear

- Substitute \( x(t) = ax_1(t) + bx_2(t) \)

\[
y(t) = \int_{-\infty}^{\infty} [ax_1(\tau) + bx_2(\tau)]h(t - \tau) \, d\tau
\]

\[
= a \int_{-\infty}^{\infty} x_1(\tau)h(t - \tau) \, d\tau + b \int_{-\infty}^{\infty} x_2(\tau)h(t - \tau) \, d\tau
\]

\[
= ay_1(t) + by_2(t)
\]

Therefore, convolution is linear.
Convolution is Time-Invariant

- Substitute \( x(t-t_0) \)

\[
\begin{align*}
w(t) &= \int_{-\infty}^{\infty} h(\tau)x((t - \tau) - t_0) \, d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)x((t - t_0) - \tau) \, d\tau \\
&= y(t - t_0)
\end{align*}
\]
Cascade of LTI Systems

\[ h(t) = h_1(t) * h_2(t) = h_2(t) * h_1(t) \]

(b)
Parallel LTI Systems

\[ h(t) = h_1(t) + h_2(t)(t) \]
Example: More complicated combinations
READING ASSIGNMENTS

- This Lecture:
  - Chapter 9, Sects. 9-6, 9-7, and 9-8

- Other Reading:
  - Ch. 9, all
  - Next Lecture: Start reading Chapter 10