Linear and Time Invariant System

• What is LTI?
• How to prove?
• Properties
  – Completely characterized by impulse response \( h[n] = T(\delta[n]) \)
  – \( y[n] = x[n] * h[n] \)
• Given \( y_1[n] = T(x_1[n]) \), \( y_2[n] = T(x_2[n]) \), can determine \( y[n] \) for combination/shifted versions of \( x_1[n] \) and \( x_2[n] \)
**FIR System**

- A special LTI system
- Described by the feed-forward difference equation
  \[ y[n] = \sum_{k=0}^{M} b_k x[n-k] \]
- Impulse response \( h[n] = b_n \) (finite duration)

**Discrete Time Convolution**

- \( y[n] = x[n] * h[n] = \sum_k h[k] x[n-k] \)
- Two types of computation
  - Sliding window approach
  - Tabular approach (synthetic polynomial multiplication)
- Relation of Length of filter (\( N_1 \)), input (\( N_2 \)), output (\( N \))
  \[ N = N_1 + N_2 - 1 \]
IIR system

- A special LTI system
- Described by the difference equation with feedback
  \[ y[n] = \sum_{k=0}^{M} b_k x[n-k] + \sum_{l=1}^{L} a_l y[n-l] \]
- Impulse response \( h[n] \) not directly related to \( \{a_l,b_k\} \)
- Generally infinite duration
- Convolution: involving generally two infinite duration signals, not easy
  - Can compute manually using the difference equation, recursively
  - Can use \( \text{filter}(b,a,x) \) to compute impulse response with \( x \) set to a delta function
  - Can use \( \text{filter}(b,a,x) \) to compute output to any input
- Z-domain representation can greatly simplify analysis

Frequency Response

- \( h[n] \leftrightarrow H(e^{jw}) = \sum h[n]e^{-jnw} \)
- \( x[n] = e^{j\omega n} \rightarrow y[n] = H(e^{j\omega}) e^{j\omega} \)
- \( X[n] = \cos(\omega n + \phi) \rightarrow y[n] = |H(e^{j\omega})| \cos(\omega n + \phi + <H(e^{j\omega})) \)
- \( H(e^{jw}) \) tells how the magnitude and phase of an input sinusoid signal is changed by the system
- If \( H(e^{jw}) = 0 \) for some \( w \), the system zeros out the input signal with frequency \( w \). (\( w \) is the null frequency)
- Low pass, high pass, bandpass concept.
  - Low pass: blurring
  - High pass: edge detection, sharpening
Z-transform

• General definition on any signal
  – \( x[n] \leftrightarrow X(z) \)

• Important property
  – \( x[n] h[n] \leftrightarrow X(z) H(z) \)
  – \( x[n-d] \leftrightarrow z^{-d} X(z) \)
  – \( a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}, \text{ for } |z|>|a| (\text{ROC}) \)

• Inverse Z-transform
  – Using partial fraction expansion to represent any rational function \( X(z) \) as sum of first order systems
  – Then using transform pair for first order system to derive \( x[n] \) from \( X(z) \)

• Using Z-transform to compute \( y[n] \)
  – \( x[n] \rightarrow X(z), Y(z)=H(z) X(z), \ Y(z) \rightarrow y[n] \)

Z-Transform of FIR/IIR System (System Function)

• Difference equation \( \leftrightarrow \) \( H(z) \leftrightarrow \) impulse response

\[
y[n] = \sum_{\ell=1}^{N} a_{\ell} y[n-\ell] + \sum_{k=0}^{M} b_{k} x[n-k] \]
\[
H(z) = \frac{\sum b_{k} z^{-k}}{1-\sum a_{\ell} z^{-\ell}}
\]

• Zeros and poles
  – Should be able to determine zeros and poles of up to 2\(^{nd}\) order systems and plot zeros and poles on z-plane
  – Can use \texttt{zplane()} to plots zeros and poles of any system
  – Use \texttt{roots()} to determine zeros of any given polynomial
  – Roots of real polynomials are either real or appear in conjugate pairs
Poles/Zeros of FIR System

- Zeros define null frequency of the filter
- Poles at $z=0$ only
- Can sketch frequency response based on zeros

11-pt RUNNING SUM $H(z)$

$$H(z) = \sum_{k=0}^{10} z^{-k}$$

$$H(z) = (1 - e^{j2\pi/11}z^{-1})(1 - e^{j4\pi/11}z^{-1}) \cdots (1 - e^{j20\pi/11}z^{-1})$$

![Diagram of 11-point running sum filter and its frequency response with no zero at $z=1$]
Real Symmetric FIR Filters

- \( h_k = h_{M-k} \)
- Generally zeros appear in quadruplets

\[
z_0, z_0^*, 1/z_0, 1/z_0^*
\]

If \( z_0 = re^{j\theta} \), then the four associated zeros are

\[
re^{j\theta}, re^{-j\theta}, \frac{1}{r}e^{j\theta}, \frac{1}{r}e^{-j\theta}
\]

Poles/Zeros of IIR System

- For real system, appear in conjugate pairs
- Impulse response of 1st and 2nd order system
  - First order with pole at \( p^n u[n] \) (exponential decay)
  - Second order with complex poles at \( p = re^{j\theta} \) and \( p^* = 2r^n \cos(\theta n) u[n] \) (decaying sinusoid)
    - \( r \) determines decay rate
    - \( \theta \) determines sinusoid freq.
Stability of IIR System

- BIBO stability definition
- Sufficient condition for BIBO stability
  - \( \sum |h[n]| < \infty \)
  - Poles inside unit circle (\(|p_k| < 1\))
- Transient and Steady State Response to Suddenly Applied Sinusoid Input
- Transient signal dies out if the system is stable

THREE DOMAINS

- **Z-TRANSFORM-DOMAIN**: poles & zeros
- **POLYNOMIALS**: \( H(z) \)
  - Use \( H(z) \) to get Freq. Response
- **TIME-DOMAIN**: 
  - \( y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k] \)
- **FREQ-DOMAIN**: 
  - \( H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 - \sum_{\ell=1}^{N} a_\ell e^{-j\omega \ell}} \)
MAPPING BETWEEN $z$ and $\omega$ on UNIT CIRCLE

- MAPPING BETWEEN $z$ and $\omega$

  $$z = e^{j\omega}$$

  - $z = 1 \iff \omega = 0$
  - $z = -1 \iff \omega = \pm \pi$
  - $z = \pm j \iff \omega = \pm \frac{1}{2} \pi$

Determine frequency response from poles and zeros

- $H(e^{j\omega}) = H(z) \mid z=e^{j\omega}$ (unit circle)
- Zeros on or near unit circle make $H(e^{j\omega})$ go to zero
- Poles on or near unit circle make $H(e^{j\omega})$ go to local maxima
Determine impulse response from poles for IIR system

- First order with pole at \( p: p^n u[n] \)
- Second order with complex poles at \( p = re^{j\theta} \) and \( p^*: r^n \cos(\theta n) u[n] \)
- Using partial fraction expansion to expand any given system function into sum of first order and second order systems and a FIR term
Logistics of Test 1 (Tues 3-5PM)

- Closed book, can bring 1 page note (double sided OK)
- Office hour: Monday 11-1PM, LC256, Tuesday 10-12AM
- Will post solution to this week’s homework