Laplace Transform: Definition and Region of Convergence

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Some slides included are extracted from lecture notes from MIT open courseware
Why do we need another transform?

- Fourier transform cannot handle large (and important) classes of signals and unstable systems, i.e. when

\[ \int_{-\infty}^{\infty} |x(t)| \, dt = \infty \]

- Laplace Transform can be viewed as an extension of the Fourier transform to allow analysis of broader class of signals and systems (including unstable systems!)
Eigen Function of LTI System

- $e^{st}$ is an eigenfunction of any LTI system
  - $s = \sigma + j\omega$ can be complex in general

\[ H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt \quad \text{(assuming this converges)} \]

- Show on the board
- $H(s)$ is the Laplace transform of $h(t)$!
The (Bilateral) Laplace Transform

\[ x(t) \leftrightarrow X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \mathcal{L}\{x(t)\} \]

\( s = \sigma + j\omega \) is a complex variable
Relation with Fourier Transform

(1) \[ X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt = \mathcal{F}\{x(t)e^{-\sigma t}\} \]

(2) A critical issue in dealing with Laplace transform is convergence:
   — \(X(s)\) generally exists only for some values of \(s\), located in what is called the region of convergence (ROC)

   \[ \text{ROC} = \{s = \sigma + j\omega \text{ so that } \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt < \infty \} \]

(3) If \(s = j\omega\) is in the ROC (i.e. \(\sigma = 0\)), then

   \[ X(s)\big|_{s=j\omega} = \mathcal{F}\{x(t)\} \]
Example 1

\[ x_1(t) = e^{-at}u(t) \]

(a - an arbitrary real or complex number)

Unstable:
- no \textbf{Fourier Transform}
- but \textbf{Laplace Transform} exists

\[ X_1(s) = \frac{1}{s + a}, \quad \text{ROC} \Rightarrow \Re\{s\} > -\Re\{a\} \]
- Derive result on board, sketch ROC for both $a>0$ and $a<0$
Example 2

\[ x_2(t) = -e^{-at} u(-t) \]

**Unstable:**
- no *Fourier Transform*
- but *Laplace Transform* exists

\[ X_2(s) = \frac{1}{s + a}, \quad \Re\{s\} < -\Re\{a\} \]

Same as \( X_1(s) \), but different ROC
Derive result on board
Example #1

\[ X_1(s) = \frac{1}{s + a}, \quad \Re\{s\} > -\Re\{a\} \]

\[ x_1(t) = e^{-at}u(t) - \text{right-sided signal} \]

Example #2

\[ X_2(s) = \frac{1}{s + a}, \quad \Re\{s\} < -\Re\{a\} \]

\[ x_2(t) = -e^{-at}u(-t) - \text{left-sided signal} \]

Note: same \( X(s) \) may correspond to different \( x(t) \) depending on ROC!
Example 3
\[ x(t) = e^{bt}u(-t) + e^{-bt}u(t) \]

\[
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow
\]

\[-\frac{1}{s-b}, \Re\{s\} < b \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \frac{1}{s+b}, \Re\{s\} > -b\]

Overlap if \( b > 0 \) \( \Rightarrow \) \( X(s) = \frac{-2b}{s^2 - b^2} \), with ROC:

What if \( b < 0 \)? \( \Rightarrow \) No overlap \( \Rightarrow \) No Laplace Transform
General trend of ROC

- ROCs are always vertical half planes or stripes, bounded by poles
- Right side signals -> ROC in right half plane
- Left side signals -> ROC in left half plane
- Double sided signals -> ROC in a central stripe, or does not exist
Some signals do not have Laplace Transforms (have no ROC)

(a) $x(t) = Ce^{-t}$ for all $t$ since $\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt = \infty$ for all $\sigma$

(b) $x(t) = e^{j\omega_0 t}$ for all $t$ 

$FT: X(j\omega) = 2\pi \delta(\omega - \omega_0)$

$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt = \int_{-\infty}^{\infty} e^{-\sigma t} dt = \infty$ for all $\sigma$

$X(s)$ is defined only in ROC; we don’t allow impulses in LTs
Finite duration signals that are absolutely integrable ->
ROC contains entire S-plane
Importance of ROC

- X(s) cannot uniquely define x(t)
- Need ROC and X(s)!
Inverse Laplace Transform

\[ X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt, \quad s = \sigma + j\omega \in \text{ROC} \]

\[ = \mathcal{F}\{x(t)e^{-\sigma t}\} \]

Fix \( \sigma \in \text{ROC} \) and apply the inverse Fourier transform

\[ x(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t} d\omega \]

\[ \Downarrow \]

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma+j\omega)t} d\omega \]

But \( s = \sigma + j\omega \) (\( \sigma \) fixed) \( \Rightarrow ds = jd\omega \)

\[ \Downarrow \]

\[ j\omega \text{ in the integral limit should be replaced by } j\infty \]

\[ x(t) = \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st} ds \]
Inverse Laplace Transforms Via Partial Fraction Expansion and Properties

Example:

\[ X(s) = \frac{s + 3}{(s + 1)(s - 2)} = \frac{A}{s + 1} + \frac{B}{s - 2} \]

\[ A = -\frac{2}{3}, \quad B = \frac{5}{3} \]

Three possible ROC’s — corresponding to three different signals:

Recall

\[ \frac{1}{s + a}, \quad \Re\{s\} < -a \leftrightarrow -e^{-at}u(-t) \text{ left-sided} \]

\[ \frac{1}{s + a}, \quad \Re\{s\} > -a \leftrightarrow e^{-at}u(t) \text{ right-sided} \]
ROC I: — Left-sided signal.
\[
x(t) = -Ae^{-t}u(-t) - Be^{2t}u(-t)
= \left[ \frac{2}{3} e^{-t} - \frac{5}{3} e^{2t} \right] u(-t) \quad \text{Diverges as } t \to -\infty
\]

ROC II: — Two-sided signal, has Fourier Transform.
\[
x(t) = Ae^{-t}u(t) - Be^{2t}u(-t)
= -\left[ \frac{2}{3} e^{-t}u(t) + \frac{5}{3} e^{2t}u(-t) \right] \quad \to 0 \text{ as } t \to \pm \infty
\]

ROC III: — Right-sided signal.
\[
x(t) = Ae^{-t}u(t) + Be^{2t}u(t)
= \left[ -\frac{2}{3} e^{-t} + \frac{5}{3} e^{2t} \right] u(t) \quad \text{Diverges as } t \to +\infty
\]
Partial Fraction Expansion

- Review
- For previous example:
  - $A = X(s) \frac{1}{(s+1)} \mid (s=-1)$
  - $B = X(s) \frac{1}{(s-2)} \mid (s=2)$
Convolution Property

\[ x(t) \quad \rightarrow \quad h(t) \quad \rightarrow \quad y(t) = h(t) \ast x(t) \]

For \[ x(t) \leftrightarrow X(s), \quad y(t) \leftrightarrow Y(s), \quad h(t) \leftrightarrow H(s) \]
Then \[ Y(s) = H(s) \cdot X(s) \]

- ROC of \( Y(s) = H(s)X(s) \): at least the overlap of the ROCs of \( H(s) \) & \( X(s) \)
- ROC could be empty if there is no overlap between the two ROCs
  E.g. \[ x(t) = e^t u(t), \text{ and } h(t) = -e^{-t} u(-t) \]
- ROC could be larger than the overlap of the two. E.g.
  \[ x(t) \ast h(t) = \delta(t) \]
proof
The System Function of an LTI System

\[ x(t) \rightarrow h(t) \rightarrow y(t) \]

\[ h(t) \leftrightarrow H(s) - \text{the system function} \]

The system function characterizes the system

\[ \downarrow \]

System properties correspond to properties of \( H(s) \) and its ROC

A first example:

System is stable \( \Leftrightarrow \int_{-\infty}^{\infty} |h(t)| dt < \infty \Leftrightarrow \) ROC of \( H(s) \) includes the \( j\omega \) axis
Sketching Fourier Transform from Pole-Zero Locations

\[ H(s) = \frac{1}{s\tau + 1} = \frac{1/\tau}{s + 1/\tau}, \quad \Re\{s\} > -\frac{1}{\tau} \]
First-Order All-Pass System

\[ H(s) = \frac{s - a}{s + a}, \quad \Re\{s\} > -a \quad (a > 0) \]

1. Two vectors have the same lengths
2. \( \angle H(j\omega) = \theta_1 - \theta_2 \)
   \[ = (\pi - \theta_2) - \theta_2 \]
   \[ = \pi - 2\theta_2 \]

\[ |H(j\omega)| = \begin{cases} 
\pi & \omega = 0 \\
\pi/2 & \omega = a \\
\approx 0 & \omega >> a 
\end{cases} \]
Readings

- Oppenheim and Willsky, Signals and Systems, Sec. 9.0—9.4 (Handout)