EE3054
Signals and Systems

Lecture 2
Linear and Time Invariant Systems

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Review of Last Lecture

- General FIR System
  \[ \{b_k\} \]
- IMPULSE RESPONSE
  \[ h[n] \]
  - FIR case: \( h[n] = b_n \)
- CONVOLUTION
  \[ y[n] = h[n] \ast x[n] \]
  - For any FIR system: \( y[n] = x[n] \ast h[n] \)
GENERAL FIR FILTER

- FILTER COEFFICIENTS \( \{b_k\} \)
- DEFINE THE FILTER

For example,

\[
y[n] = \sum_{k=0}^{M} b_k x[n - k]
\]

\[
b_k = \{3, -1, 2, 1\}
\]

\[
y[n] = \sum_{k=0}^{3} b_k x[n - k]
\]

\[
= 3x[n] - x[n - 1] + 2x[n - 2] + x[n - 3]
\]
GENERAL FIR FILTER

- SLIDE a Length-L WINDOW over x[n]
- When h[n] is not symmetric, needs to flip h(n) first!

\[ y[n] = \sum_{k=0}^{M} b_k x[n - k] \]

\( M \)-th Order FIR Filter Operation (Causal)
Input: $x[n] = (1.02)^n + \cos\left(\frac{2\pi n}{8} + \frac{\pi}{4}\right)$ for $0 \leq n \leq 40$
Unit Impulse Signal

- \( x[n] \) has only one NON-ZERO VALUE

\[
\delta[n] = \begin{cases} 
1 & n = 0 \\
0 & n \neq 0 
\end{cases}
\]

UNIT-IMPULSE
4-pt Avg Impulse Response

\[ y[n] = \frac{1}{4} (x[n] + x[n - 1] + x[n - 2] + x[n - 3]) \]

\[ \delta[n] \] “READS OUT” the FILTER COEFFICIENTS

\[ h[n] = \{..., 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, ...\} \]

“h” in \( h[n] \) denotes Impulse Response

NON-ZERO When window overlaps \( \delta[n] \)
What is Impulse Response?

- Impulse response is the output signal when the input is an impulse.
- Finite Impulse Response (FIR) system:
  - Systems for which the impulse response has finite duration.
- For FIR system, impulse response = Filter coefficients:
  - \( h[k] = b_k \)
  - Output = \( h[k] \ast \text{input} \)
**FIR IMPULSE RESPONSE**

- Convolution = Filter Definition
- Filter Coeffs = Impulse Response

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n &lt; 0$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>$M$</th>
<th>$M + 1$</th>
<th>$n &gt; M + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[n] = \delta[n]$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y[n] = h[n]$</td>
<td>0</td>
<td>$b_0$</td>
<td>$b_1$</td>
<td>$b_2$</td>
<td>$b_3$</td>
<td>...</td>
<td>$b_M$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ y[n] = \sum_{k=0}^{M} b_k x[n-k] \]

\[ y[n] = \sum_{k=0}^{M} h[k] x[n-k] \]
Convolution Operation

- Flip \( h[n] \)
- SLIDE a Length-L WINDOW over \( x[n] \)

\[
y[n] = \sum_{k=0}^{M} h[k] x[n-k]
\]

M-th Order FIR Filter Operation (Causal)
More on signal ranges and lengths after filtering

- Input signal from 0 to N-1, length = L1 = N
- Filter from 0 to M, length = L2 = M+1
- Output signal?
  - From 0 to N+M-1, length = L3 = N+M = L1+L2-1
DCONVDEMO: MATLAB GUI

Signal Axis:
- $x[k]
- h[n-k]

Multiplication Axis:
- $x[k]h[n-k]

Convolution Axis:
- $y[n] = \sum x[k]h[n-k]$
- Go through the demo program for different types of signals
- Do an example by hand
  - Rectangular * rectangular
  - Step function * rectangular
### CONVOLUTION via Synthetic Polynomial Multiplication

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>x[n]</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[n]</td>
<td>3</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[0]x[n-0]</td>
<td>6</td>
<td>12</td>
<td>18</td>
<td>12</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[1]x[n-1]</td>
<td>-2</td>
<td>-4</td>
<td>-6</td>
<td>-4</td>
<td>-2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[2]x[n-2]</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>8</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[3]x[n-3]</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>y[n]</td>
<td>6</td>
<td>10</td>
<td>18</td>
<td>16</td>
<td>18</td>
<td>12</td>
<td>8</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
Convolution via Synthetic Polynomial Multiplication

- More example
MATLAB for FIR FILTER

- \( yy = \text{conv}(bb, xx) \)
  - VECTOR \( bb \) contains Filter Coefficients
  - DSP-First: \( yy = \text{firfilt}(bb, xx) \)

- FILTER COEFFICIENTS \( \{b_k\} \)

\[
y[n] = \sum_{k=0}^{M} b_k x[n - k]
\]

conv2 ()
for images
POP QUIZ

- FIR Filter is “FIRST DIFFERENCE”
  - $y[n] = x[n] - x[n-1]$
- INPUT is “UNIT STEP”

Find $y[n]$

$y[n] = u[n] - u[n-1] = \delta[n]$
SYSTEM PROPERTIES

- MATHEMATICAL DESCRIPTION
- TIME-INVARIANCE
- LINEARITY
- CAUSALITY
  - “No output prior to input”
TIME-ININVARIANCE

IDEA:
- “Time-Shifting the input will cause the same time-shift in the output”

EQUIVALENTLY,
- We can prove that
  - The time origin (n=0) is picked arbitrary
TESTING Time-Invariance

Figure 5.16 Testing time-invariance property by checking the interchange of operations.

\[ w[n] \text{ must equal } y[n - n_0] \text{ when the system is time invariant} \]
Examples of systems that are time invariant and non-time invariant
LINEAR SYSTEM

- LINEARITY = Two Properties
- SCALING
  - “Doubling x[n] will double y[n]”
- SUPERPOSITION:
  - “Adding two inputs gives an output that is the sum of the individual outputs”
Figure 5.17  Testing linearity by checking the interchange of operations.
Examples systems that are linear and non-linear
LTI SYSTEMS

- LTI: Linear & Time-Invariant
- Any FIR system is LTI
- Proof!
LTI SYSTEMS

- COMPLETELY CHARACTERIZED by:
  - IMPULSE RESPONSE $h[n]$
  - CONVOLUTION: $y[n] = x[n]*h[n]$
    - The “rule” defining the system can ALWAYS be re-written as convolution
  - FIR Example: $h[n]$ is same as $b_k$
Proof of the convolution sum relation
- by representing $x(n)$ as sum of $\delta(n-k)$, and use LTI property!
Properties of Convolution

- Convolution with an Impulse
- Commutative Property
- Associative Property
Convolution with Impulse

- $x[n] * \delta[n] = x[n]$
- $x[n] * \delta[n-k] = x[n-k]$
- Proof
Commutative Property

- $x[n]*h[n]=h[n]*x[n]$
- Proof
Associative Property

- \((x[n] \cdot y[n]) \cdot z[n] = x[n] \cdot (y[n] \cdot z[n])\)
- Proof
HARDWARE STRUCTURES

- INTERNAL STRUCTURE of “FILTER”
  - WHAT COMPONENTS ARE NEEDED?
  - HOW DO WE “HOOK” THEM TOGETHER?
- SIGNAL FLOW GRAPH NOTATION

\[ y[n] = \sum_{k=0}^{M} b_k x[n - k] \]
HARDWARE ATOMS

- Add, Multiply & Store

\[ y[n] = \sum_{k=0}^{M} b_k x[n - k] \]

\[ y[n] = x_1[n] + x_2[n] \]

\[ y[n] = \beta x[n] \]

\[ y[n] = x[n - 1] \]
**FIR STRUCTURE**

- Direct Form

\[ y[n] = \sum_{k=0}^{M} b_k x[n - k] \]

**Figure 5.13**  Block-diagram structure for the \( M \)th order FIR filter.
FILTER AS BUILDING BLOCKS

- BUILD UP COMPLICATED FILTERS
  - FROM SIMPLE MODULES
  - Ex: FILTER MODULE MIGHT BE 3-pt FIR
- Is the overall system still LTI? What is its impulse response?
CASCADE SYSTEMS

- Does the order of $S_1$ & $S_2$ matter?
  - NO, LTI SYSTEMS can be rearranged !!!
  - WHAT ARE THE FILTER COEFFS? $\{b_k\}$

![Diagram of a Cascade of Two LTI Systems](image)

**Figure 5.19** A Cascade of Two LTI Systems.
- proof
Is the cascaded system LTI?

What is the impulse response of the overall system?
CASCADE SYSTEMS

\[ h[n] = h_1[n] * h_2[n]! \]

- Proof on board

**Figure 5.19** A Cascade of Two LTI Systems.
Does the order matter?

**Figure 5.19** A Cascade of Two LTI Systems.

**Figure 5.20** Switching the order of cascaded LTI systems.
proof
Example

- Given impulse responses of two systems, determine the overall impulse response
Parallel Connections

\[ h[n] = + \]

\[ x[n] \rightarrow h_1[n] \rightarrow + \rightarrow y[n] \]

\[ h_2[n] \rightarrow + \rightarrow y[n] \]

\[ h[n] = ? \]
Parallel and Cascade

\[ h[n] = h_1[n] + h_2[n] + h_3[n] \]

Diagram:

- \( x[n] \) to \( h_1[n] \) to \( h_2[n] \) to \( + \) to \( y[n] \)
- \( x[n] \) to \( h_3[n] \) to \( + \) to \( y[n] \)

\( h[n] = ? \)
Summary of This Lecture

- Properties of linear and time invariant systems
- Any LTI system can be characterized by its impulse response $h[n]$, and output is related to input by the convolution sum: $y[n] = x[n] * h[n]$
- Properties of convolution
- Computation of convolution revisited
  - Sliding window
  - Synthetic polynomial multiplication
- Block diagram representation
  - Hardware implementation of one FIR
  - Connection of multiple FIR
    - Know how to compute overall impulse response
READING ASSIGNMENTS

- This Lecture:
  - Chapter 5, Sections 5-5 --- 5-9