EE3054
Signals and Systems

Frequency Response of Continuous Time LTI Systems

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Most of the slides included are extracted from lecture presentations prepared by McClellan and Schafer

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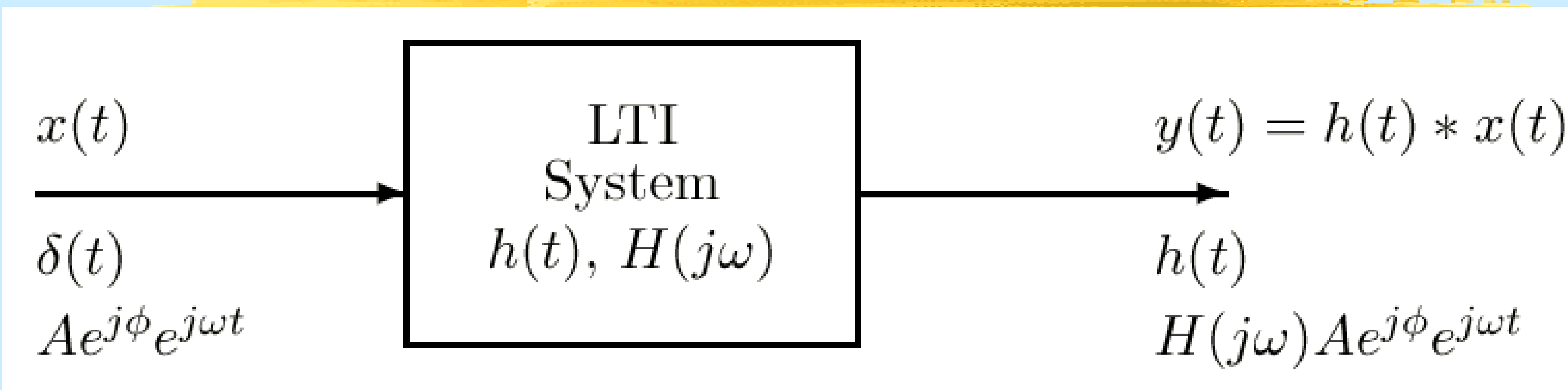
LECTURE OBJECTIVES

- Review of convolution
  - **THE** operation for **LTI** Systems
- Complex exponential input signals
  - **Frequency Response**
  - Cosine signals
    - Real part of complex exponential
- Fourier Series thru $H(j\omega)$
  - These are Analog Filters

LTI Systems



- Convolution defines an LTI system

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Response to a complex exponential gives frequency response $H(j\omega)$

Thought Process #1

- SUPERPOSITION (Linearity)
  - Make x(t) a weighted sum of signals
  - Then y(t) is also a sum—same weights
    - But DIFFERENT OUTPUT SIGNALS usually
- Use SINUSOIDS
    - "SINUSOID IN GIVES SINUSOID OUT"
  - Make x(t) a weighted sum of sinusoids
  - Then y(t) is also a sum of sinusoids
    - Different Magnitudes and Phase
- LTI SYSTEMS: Sinusoidal Response

Thought Process #2

- SUPERPOSITION (Linearity)
  - Make x(t) a weighted sum of signals
- **Use SINUSOIDS**
  - **Any x(t) = weighted sum of sinusoids**
  - **HOW? Use FOURIER ANALYSIS INTEGRAL**
    - **To find the weights from x(t)**
- LTI SYSTEMS:
  - Frequency Response changes each sinusoidal component

Complex Exponential Input

$$x(t) = Ae^{j\varphi}e^{j\omega t} \mapsto y(t) = H(j\omega)Ae^{j\varphi}e^{j\omega t}$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)Ae^{j\varphi}e^{j\omega(t-\tau)}d\tau$$

$$y(t) = \left(\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \right) Ae^{j\varphi}e^{j\omega t}$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

Frequency Response

When does $H(j\omega)$ Exist?

- When is $|H(j\omega)| < \infty$?

$$|H(j\omega)| = \left| \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \right| \le \int_{-\infty}^{\infty} |h(\tau)| \left| e^{-j\omega\tau} \right| d\tau$$

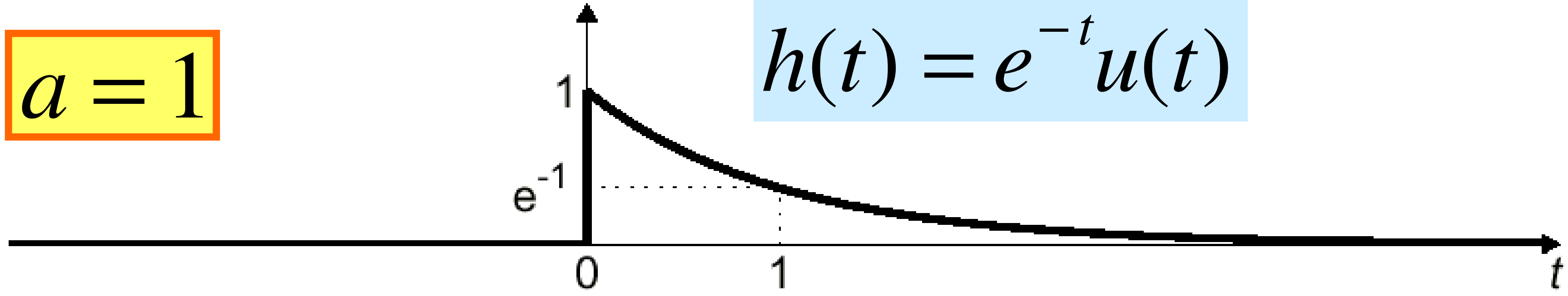
$$|H(j\omega)| \le \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

- Thus the frequency response exists if the LTI system is a **stable** system.

$$h(t) = e^{-at}u(t) \Leftrightarrow H(j\omega) = \frac{1}{a + j\omega}$$

- Suppose that h(t) is:

$a = 1$



$$h(t) = e^{-t}u(t)$$

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-j\omega\tau}d\tau = \int_{0}^{\infty} e^{-(a+j\omega)\tau}d\tau$$

$a > 0$

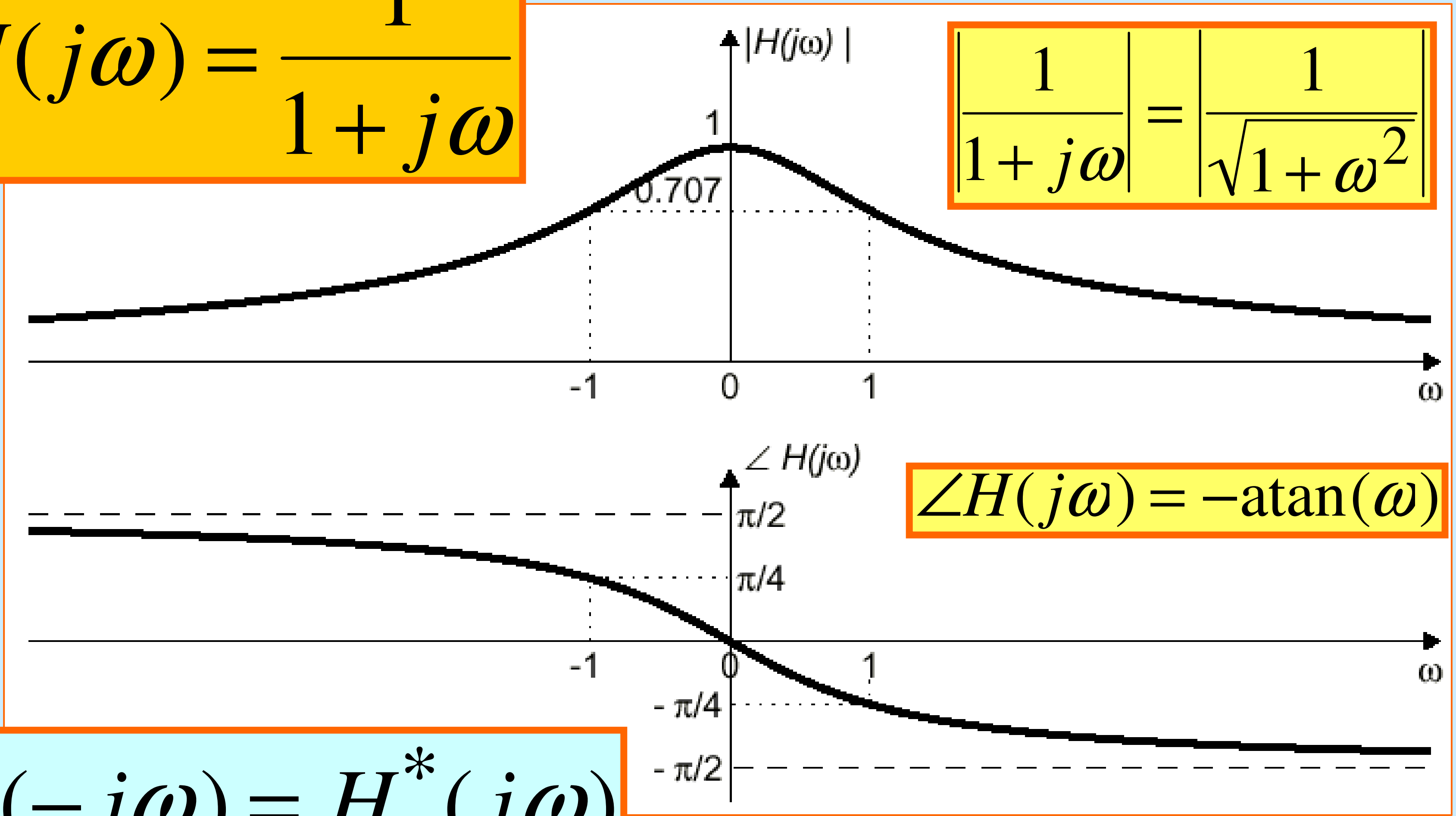
$$H(j\omega) = \left.\frac{e^{-(a+j\omega)\tau}}{-(a+j\omega)}\right|_0^\infty = \left.\frac{e^{-a\tau}e^{-j\omega\tau}}{-(a+j\omega)}\right|_0^\infty = \frac{1}{a+j\omega}$$

Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{1 + j\omega}$$

$$\left|\frac{1}{1 + j\omega}\right| = \left|\frac{1}{\sqrt{1 + \omega^2}}\right|$$

$$\angle H(j\omega) = -\text{atan}(\omega)$$



$$H(-j\omega) = H^*(j\omega)$$

Freq Response of Integrator?

- Impulse Response
  - h(t) = u(t)
- NOT a Stable System
  - Frequency response *H(jω)* does NOT exist

$$y(t) = x(t) * h(t) = \int_{-\infty}^{t} x(\tau)d\tau \quad \text{(integrator!)}$$

$$h(t) = e^{-at}u(t) \Leftrightarrow H(j\omega) = \frac{1}{a + j\omega} \rightarrow \frac{1}{j\omega}?$$

Need another term

"Leaky" Integrator (a is small)

Cannot build a perfect Integral

$$a \rightarrow 0$$

Example: Rectangular pulse

- h(t)=u(t)-u(t-10)

$$y(t) = x(t) * h(t) = \int_{t-10}^{t} x(\tau)d\tau$$

(integrator over a short interval in the past or average!)

- Show H(jw) is a sinc function

Ideal Delay:

$$y(t) = x(t - t_d)$$

$$H(j\omega) = \int_{-\infty}^{\infty} \delta(\tau - t_d) e^{-j\omega\tau} d\tau = e^{-j\omega t_d}$$

$$H(j\omega) = e^{-j\omega t_d}$$

$$x(t) = e^{j\omega t} \mapsto$$

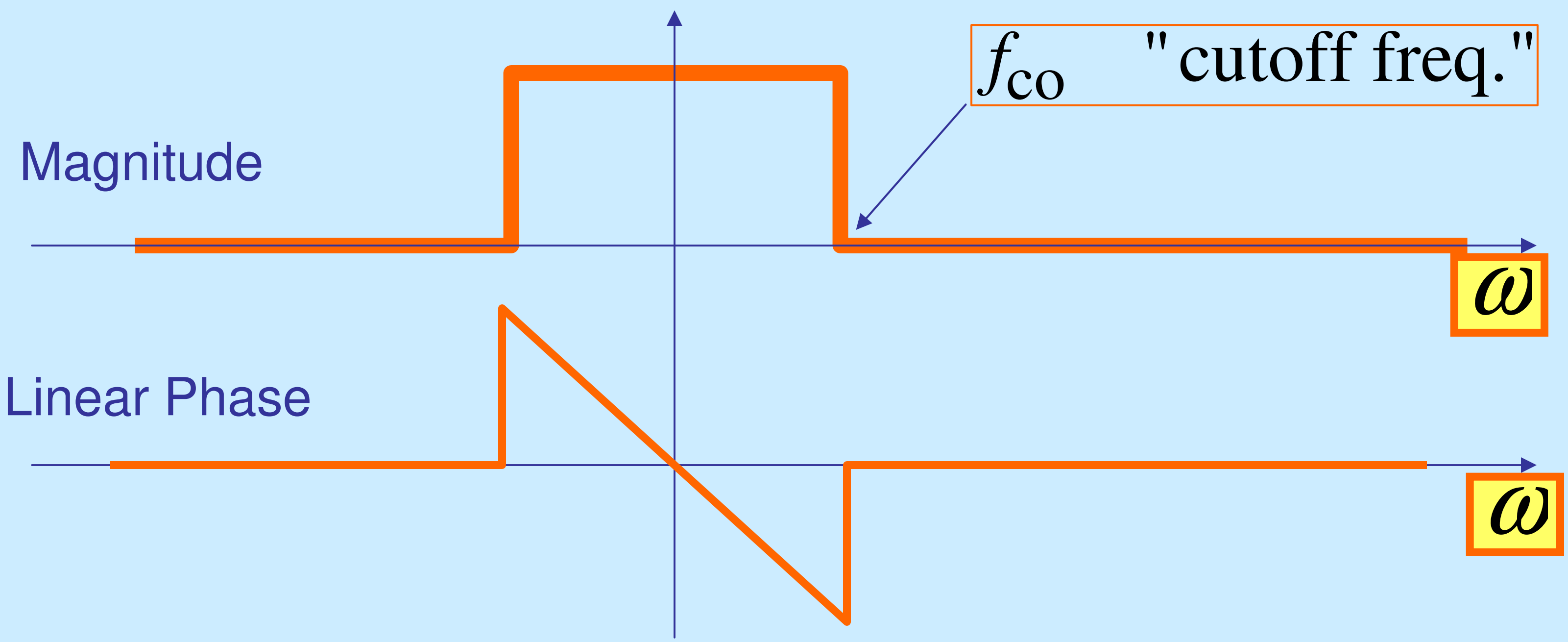
$$y(t) = e^{j\omega(t - t_d)} = \left(e^{-j\omega t_d}\right) e^{j\omega t}$$

$$H(j\omega)$$

- Delay system is All-pass with linear phase

Ideal Lowpass Filter w/ Delay

$$H_{LP}(j\omega) = \begin{cases} e^{-j\omega t_d} & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



Example: Ideal Low Pass

$$H_{LP}(j\omega) = \begin{cases} e^{-j3\omega} & |\omega| < 2 \\ 0 & |\omega| > 2 \end{cases}$$

$$x(t) = 10e^{j\pi/3}e^{j1.5t} \mapsto \quad y(t) = H(j1.5)10e^{j\pi/3}e^{j1.5t}$$

$$y(t) = \left(e^{-j4.5}\right)10e^{j\pi/3}e^{j1.5t} = 10e^{j\pi/3}e^{j1.5(t-3)}$$

Cosine Input

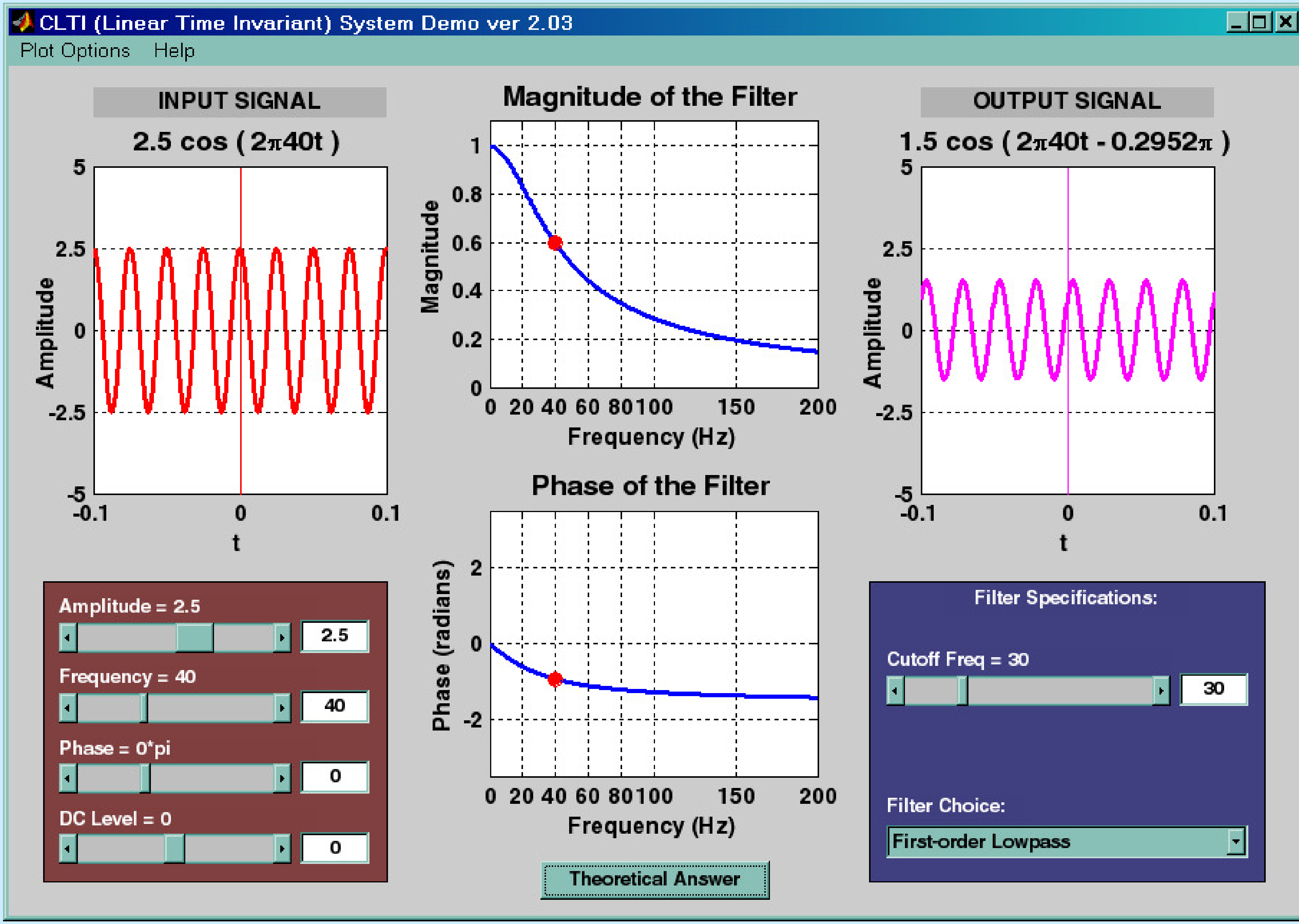
$$x(t) = A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

$$y(t) = H(j\omega_0)\frac{A}{2}e^{j\phi}e^{j\omega_0 t} + H(-j\omega_0)\frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

Since $H(-j\omega_0) = H^*(j\omega_0)$

$$y(t) = A\left|H(j\omega_0)\right|\cos(\omega_0 t + \phi + \angle H(j\omega_0))$$

Sinusoid in Gives Sinusoid out



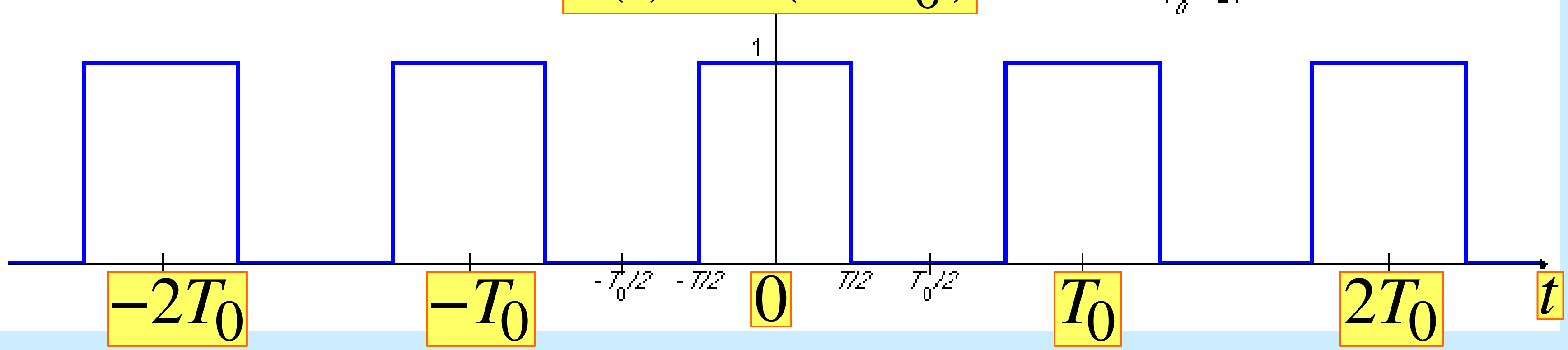
Review Fourier Series

- ANALYSIS
  - Get representation from the signal
  - Works for PERIODIC Signals
- Fourier Series
  - INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

General Periodic Signals

$$x(t) = x(t + T_0)$$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

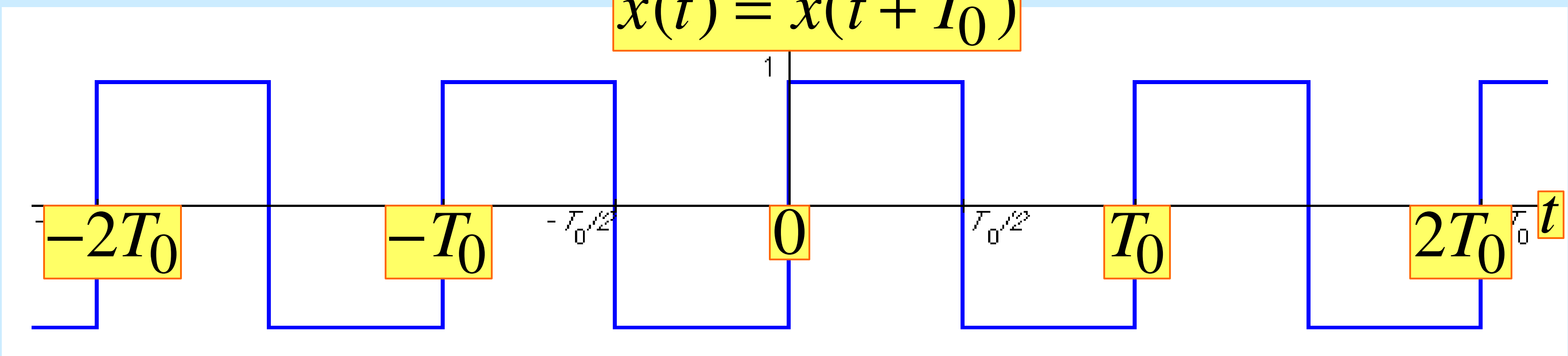
Fundamental Freq.

$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

Fourier Analysis

Square Wave Signal

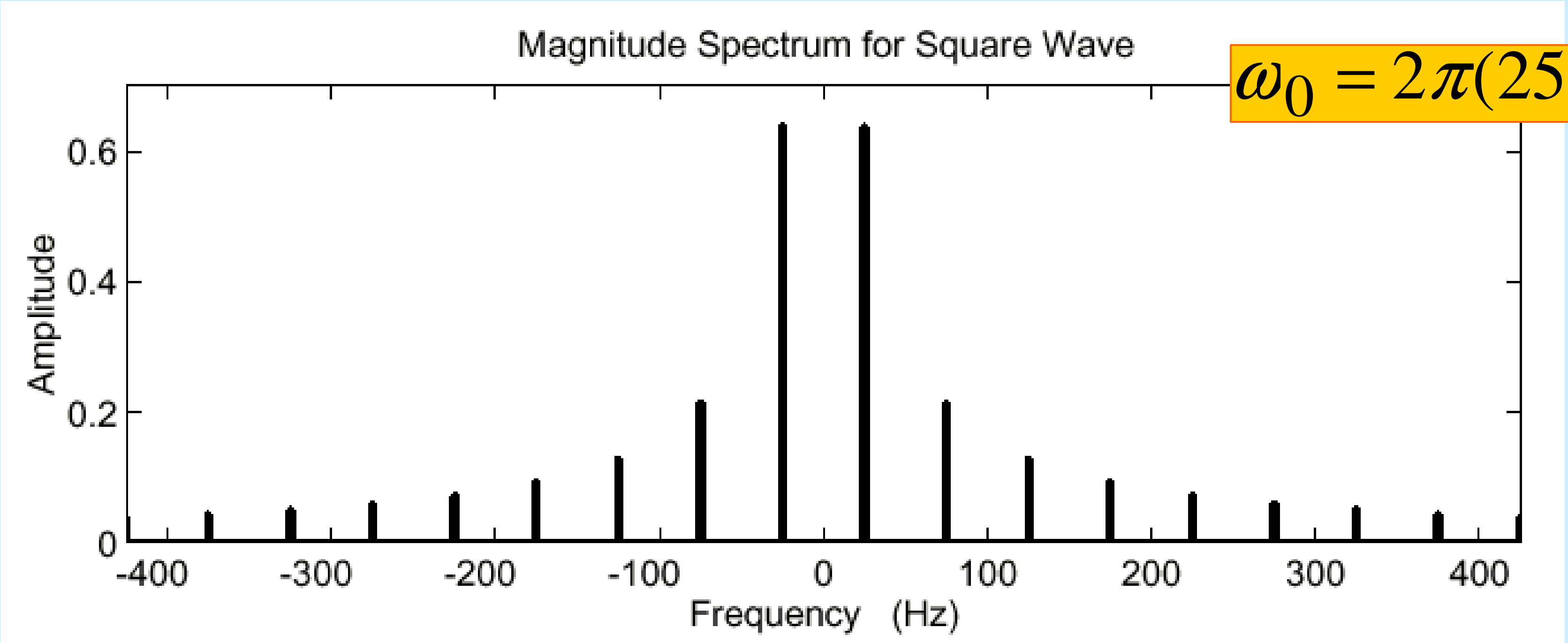


$$a_k = \frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0}(-1)e^{-j\omega_0 kt}dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1-e^{-j\pi k}}{j\pi k}$$

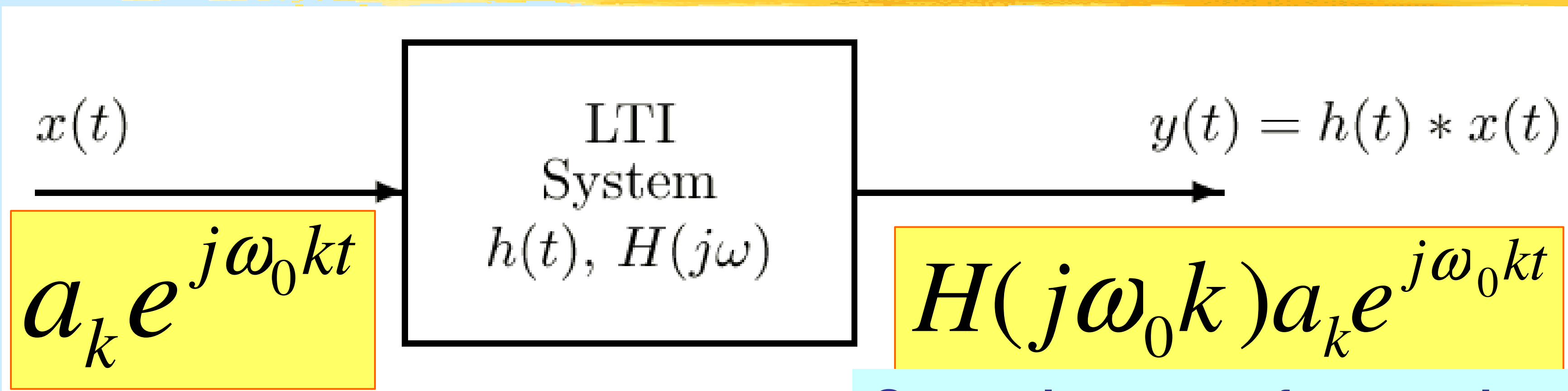
Spectrum from Fourier Series

$$a_k = \frac{1 - e^{-j\pi k}}{j\pi k} = \begin{cases} \dfrac{2}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = 0, \pm 2, \pm 4, \ldots \end{cases}$$



$\omega_0 = 2\pi(25)$

LTI Systems with Periodic Inputs



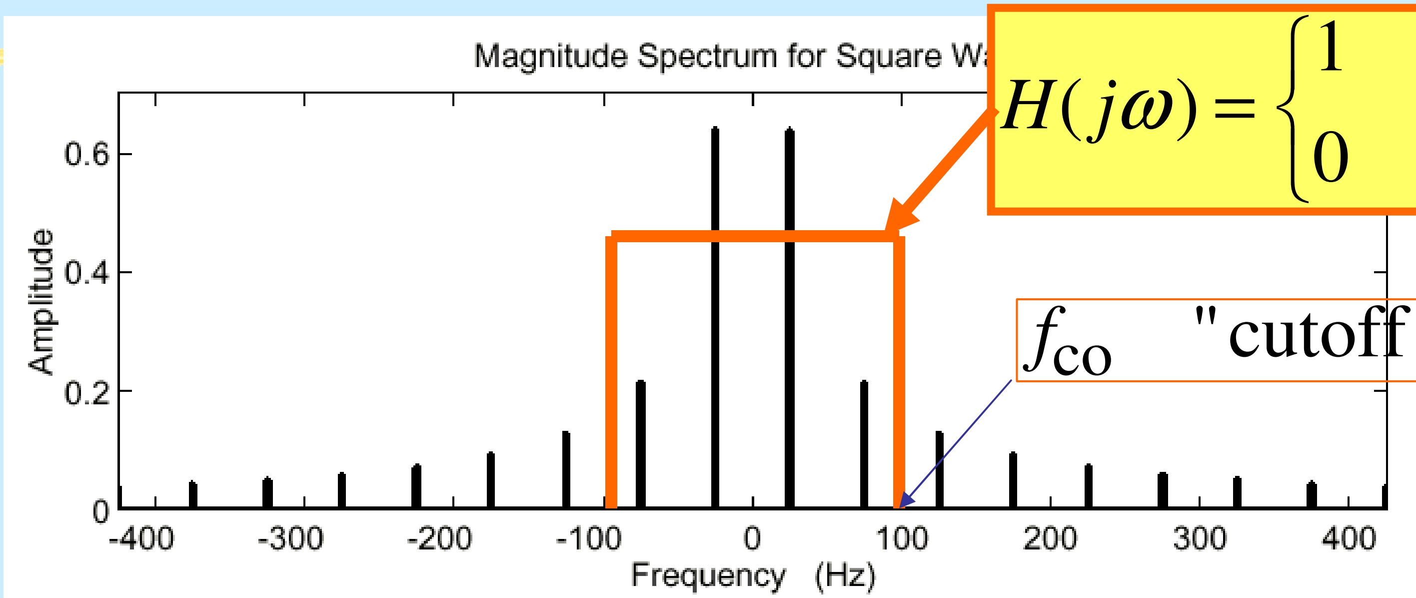
Output has *same* frequencies

- By superposition,

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 k t} = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 k t}$$

$$b_k = a_k H(j\omega_0 k)$$

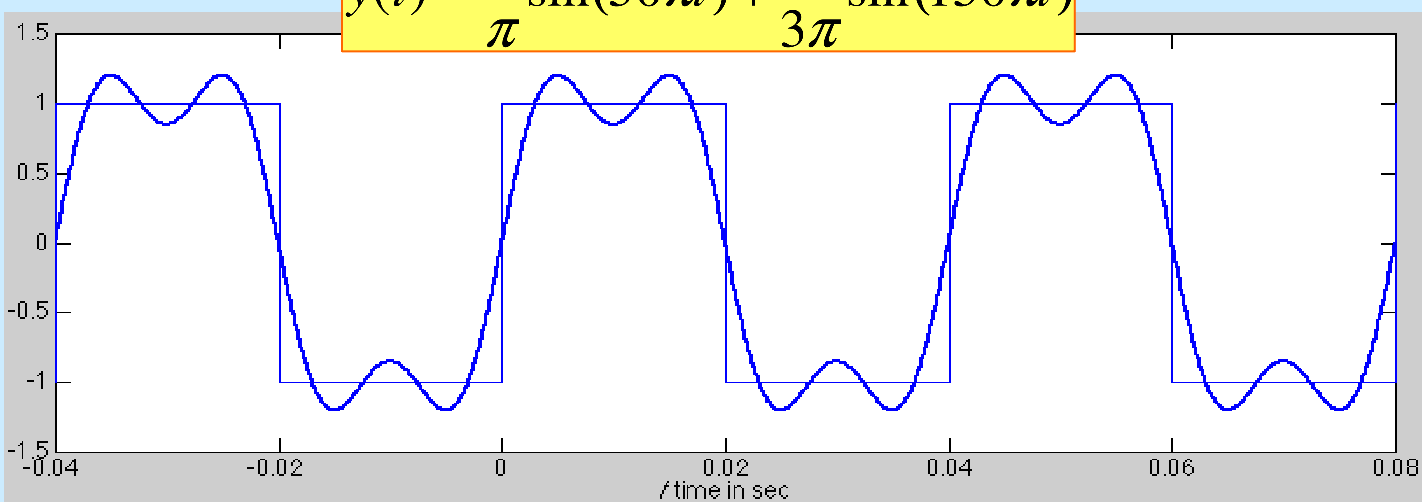
Ideal Lowpass Filter (100 Hz)



$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$

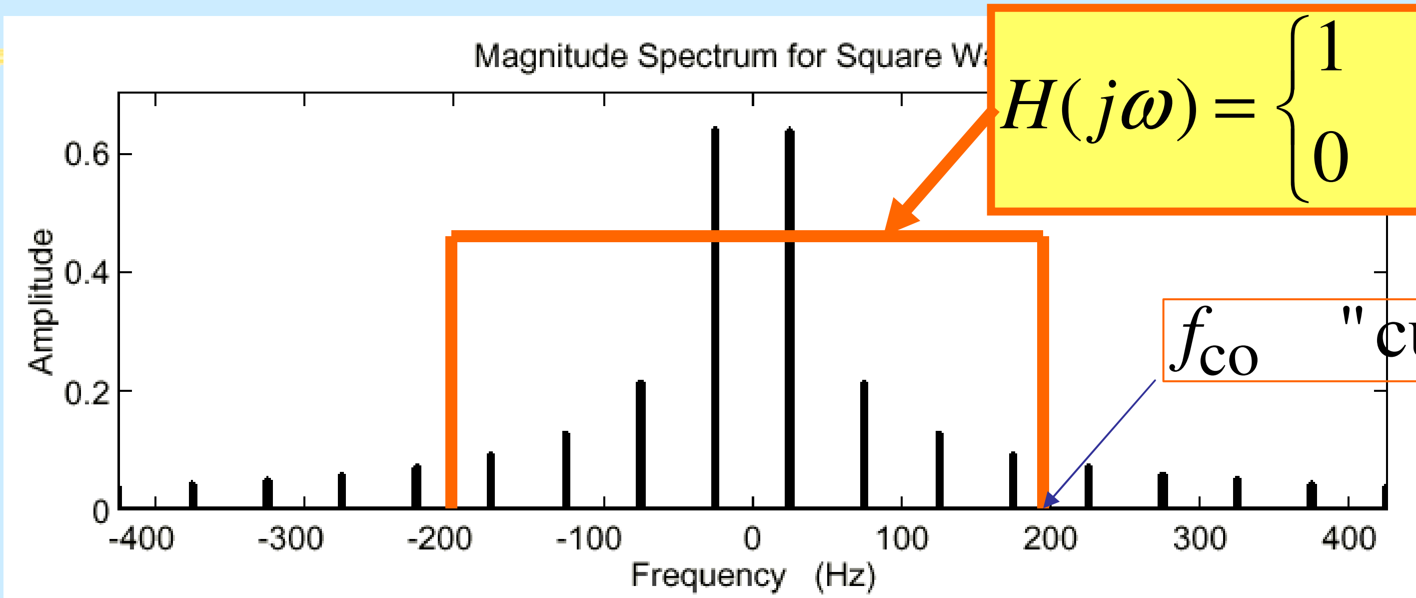
f_{co} "cutoff freq."

$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$

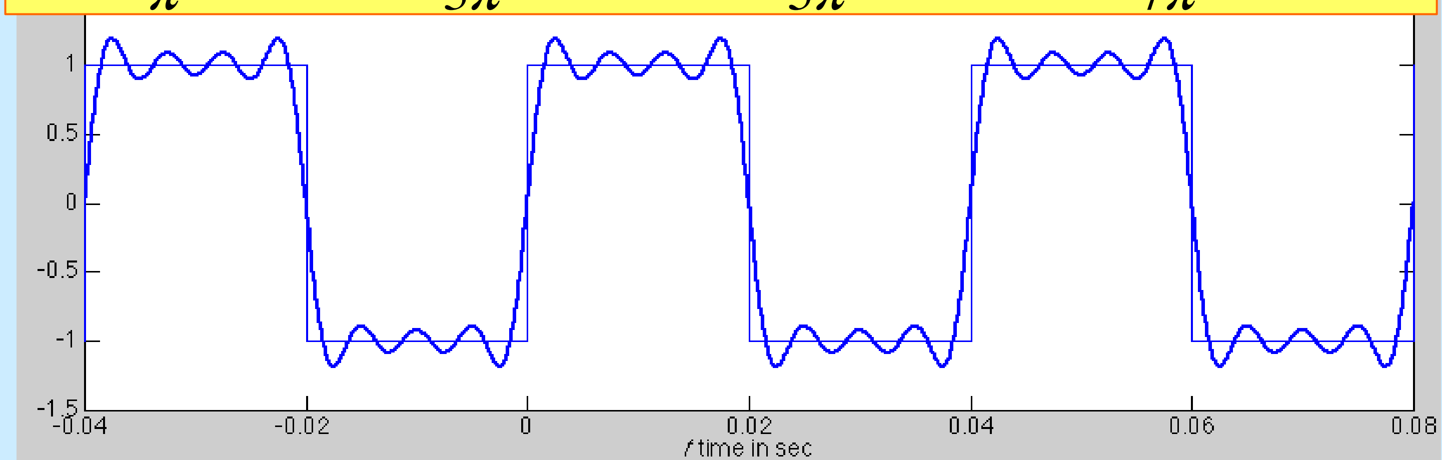


Ideal Lowpass Filter (200 Hz)

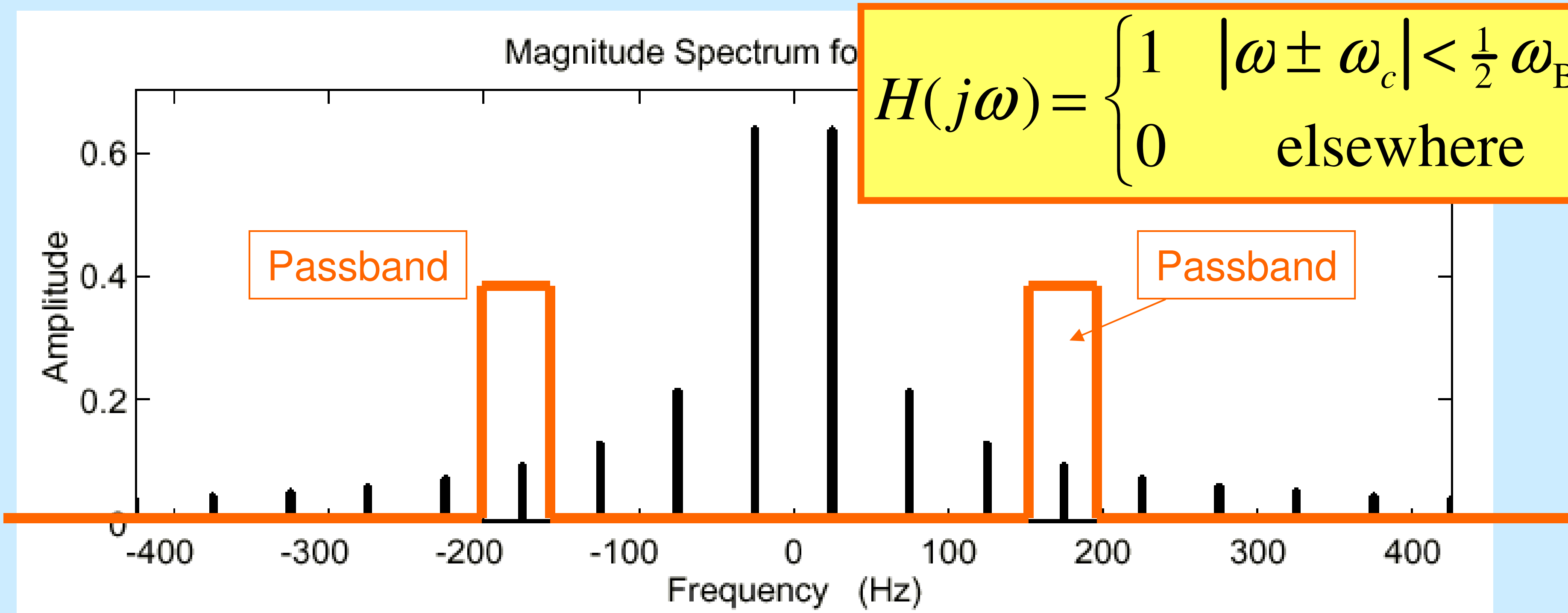
$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t) + \frac{4}{5\pi}\sin(250\pi t) + \frac{4}{7\pi}\sin(350\pi t)$$



Ideal Bandpass Filter



$$H(j\omega) = \begin{cases} 1 & \left|\omega \pm \omega_c\right| < \tfrac{1}{2}\omega_B \\ 0 & \text{elsewhere} \end{cases}$$

What is the ouput signal ?

$$y(t) = \tfrac{2}{j7\pi} e^{j2\pi(175)t} - \tfrac{2}{j7\pi} e^{-j2\pi(175)t} = \frac{4}{7\pi}\cos\left(2\pi(175)t - \tfrac{1}{2}\pi\right)$$

What will be the output if the filter in high-pass?

Ideal Delay

$$H(j\omega) = e^{-j\omega t_d}$$

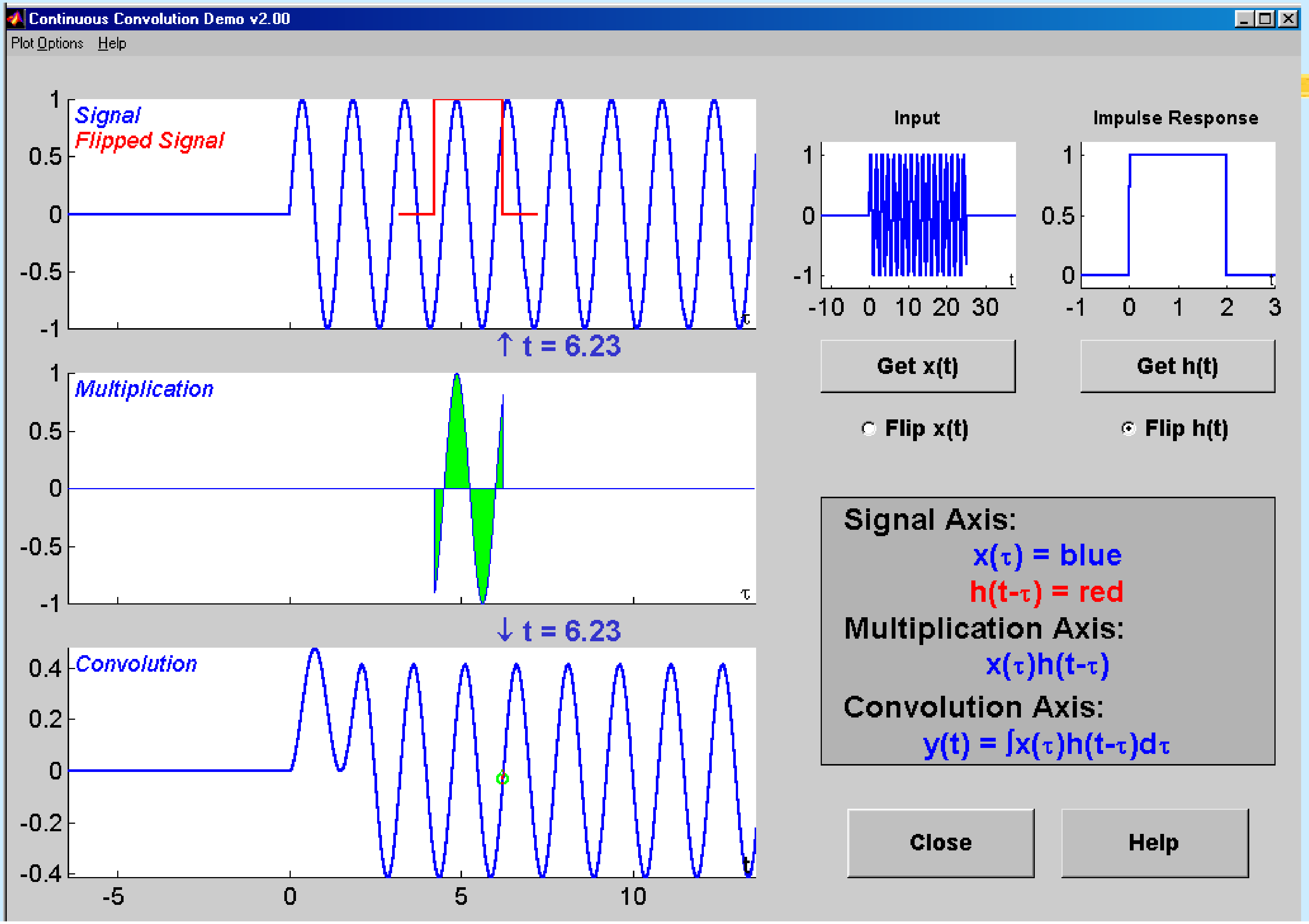
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t} \mapsto y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 k t}$$

$$b_k = a_k H(j\omega_0 k) = a_k e^{-j\omega_0 k t_d}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{-j\omega_0 k t_d} e^{j\omega_0 k t} = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k (t - t_d)}$$

$$\therefore \quad y(t) = x(t - t_d)$$

Convolution GUI: Sinusoid



Transient and Steady State Response

- Similar to discrete case

READING ASSIGNMENTS

- This Lecture:
  - Chapter 10, all