On the Performance Analysis of Cooperative Space–Time Coded Systems

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Abstract—Cooperative coding is a way to exploit the diversity provided by user cooperation in wireless networks. In this paper we provide performance analysis of cooperative space–time coding, which arises when mobiles have multiple antennas. We perform an asymptotic analysis in order to determine the achieved diversity order through cooperative space–time coding for various inter–user channel qualities. In addition, we derive tight bounds on the performance of cooperative space–time codes. Numerical examples are presented to demonstrate the results.

I. INTRODUCTION

Recently, it has been shown that user–cooperation [1], [2], [3], [4], represents an effective way of introducing diversity in wireless networks. In user–cooperation, each user aims to find a "partner", whenever possible. Each of the two partners is responsible for transmitting not only their own information, but also the information of their partner, which they receive and detect. Effectively, spatial diversity is achieved through the use of the partner’s antennas. However, this is complicated by the fact that the inter–user channel is noisy.

Figure 1 illustrates the use of user–cooperation in an ad–hoc network and in a network with a fixed access point, which may be either a local area network (LAN) or a cellular system. In the first example user $M_1$ cooperates with user $M_2$ to transmit its message to user $M_3$. Similarly, user $M_2$ cooperates with user $M_1$ to transmit its message to user $M_4$. On the other hand in a network with a fixed access point both user $M_1$ and user $M_2$ cooperate to transmit their signals to the access point (destination).

![Diagram of user-cooperation diversity in an ad-hoc network and in a network with an access point.](image)

Information theoretic studies, illustrating the benefits of cooperation can be found in [1], [4]. The problem of designing channel codes suitable for user–cooperation has been considered in [5]. The authors developed a cooperation protocol and considered cooperative coding for single–input single–output systems, based on rate compatible punctured convolutional codes. In a different approach, we recently demonstrated that the block fading channel model is appropriate for cooperative coding. We designed channel codes that can fully exploit the diversity gains of user cooperation for both single and multiple antenna systems [7], [8], [9].

In this paper, we assume the partnering mobiles and their destinations have multiple antennas. We provide an asymptotic error analysis and tight error bounds illustrating the diversity and coding gains of space–time coded cooperation. The cooperation scheme we investigate is based on a time–division system of Figure 2.

![Time-division channel allocations: (a) simultaneous direct transmission, (b) orthogonal direct transmission and (c) orthogonal cooperative diversity transmission.](image)

When cooperation is employed, $N/2$ channel uses for mobile $M_1$ are divided into two segments. This leads to $N/4$ channel uses in which user $M_1$ transmits to its partner $M_2$ and the destination, and $N/4$ channel uses in which the partner, $M_2$, relays the information of $M_1$ to the destination. Since $M_1$ and $M_2$ can be assumed to have independent fading levels towards the destination of mobile $M_1$, an overall block fading model is appropriate for the information signal of $M_1$. Similar observations can be made for $M_2$ as well. Therefore, additional diversity can be obtained through coding in time, which is provided by the cooperating mobile’s antennas. However, the fact that mobiles have a noisy and faded inter–user channel complicates the problem.

It is of interest to analyze the performance of a cooperative system and determine the achieved diversity order based on the inter–user channel quality. This can be achieved by performing asymptotic analysis at high signal–to–noise ratios. On the other hand, although such a study is useful in determining the diversity achieved by the cooperative coded system, there is still a need for the develop-
ment of tight performance bounds. This provides us with a method of easily predicting the system performance at low probability of error and comparing the performance of various codes in a short amount of time. Therefore, we describe a technique for obtaining tight bounds on the frame error probability for cooperative space-time trellis coded systems. We demonstrate that this approach provides useful bound on the frame error probability, and it accurately captures the diversity the cooperative space-time coded system achieves for various qualities of the inter-user channel.

The paper is organized as follows. In the next section we present the system model, establish notation and describe the protocols suitable for user–cooperation. Section III considers the performance analysis of the cooperative space–time coded system. We provide an asymptotic analysis, in order to study the achieved diversity order, for various inter-user channel qualities. We also derive performance bounds on the frame error probability, by expurgating the standard union bound for space-time trellis codes and using the limiting before averaging technique. Numerical examples are presented in section IV. We conclude with section V.

II. THE SYSTEM MODEL

We consider a mobile communication system, and focus on the cooperation between two users, $M_1$ and $M_2$, where user $M_i$ has $n_i$ antennas, $i = 1, 2$. Even though it is possible to investigate the scenario in which mobiles have different destinations, without loss of generality, we assume $M_1$ and $M_2$ communicate with a common destination which has $m$ antennas. The information bits are encoded by a channel encoder. The coded bits are multiplexed to arrange them accordingly in order to achieve the maximum possible diversity in the case of user cooperation. The coded and multiplexed bits are passed through a serial to parallel converter, and are mapped to a particular signal constellation. The output of the modulator is a signal $c_{i,b}^l$ that is transmitted in the $l$th time slot, $l = 1, \ldots, L$, of the fading block $b$, $b = 1, \ldots, B$, using transmit antenna $i$, $i = 1, \ldots, n$. We focus on the case when $B = 2$. All signals are transmitted simultaneously, each from a different transmit antenna, and all signals have the same transmission period $T$. The block diagram of the transmitter for each of the mobiles is given in Figure 3.

![Fig. 3. The block diagram of the transmitter.](image)

The discrete time received signal by antenna $j$, $j = 1, 2, \ldots, m$, at the destination, is given by

$$r_{l,j} = \sum_{i=1}^{n} a_{i,b}^l c_{i,b}^l + \eta_{i,b}^l, \quad l = 1, \ldots, L, \quad b = 1, \ldots, B,$$

where the noise samples $\eta_{i,b}^l$ are modeled as independent samples of a zero mean complex Gaussian random variable with variance $N_0/2$ per dimension. The coefficient $a_{i,b}^l$ is the path gain from transmit antenna $i$, $1 \leq i \leq n$, to receive antenna $j$, $1 \leq j \leq m$. We assume frequency non-selective quasi-static fading, i.e., the path gains are constant during the transmission of any given user, but are independent from user to user, hence the overall block fading assumption. We assume perfect channel state information at the receiver.

The block fading channel model (see [6] and references therein) is motivated by the fact that in many wireless systems, the coherence time of the channel is much longer than one symbol interval, resulting in adjacent symbols being affected by the same fading value. This model assumes that a codeword of length $N = BL$ symbols spans $B$ blocks of length $L$. These $N$ symbols are referred to as frame. In practice, each block of length $L$ will experience independent fading, provided we have sufficient separation in time, in frequency, or as in this case in space.

A. Cooperation Protocols

In this section, we present the protocols that the mobile users can utilize during cooperation. We first consider a fixed decode-and-forward protocol where the cooperation takes place all the time. This protocol is similar to that in [4], [5]. Then, we propose an adaptive protocol where the cooperation between the mobiles takes place only when it is necessary, resulting in an increase of the users data rate.

A.1 Fixed Decode–and–Forward Protocol

For decode-and-forward transmission, the mobiles partner receives the signal transmitted by the mobile and decodes it. Then it performs a CRC check [12] to determine whether the received sequence matches the actual transmitted information sequence. If so, the partner re–encodes the data to obtain the best possible code (in combination with the original mobiles code) which achieves maximum possible diversity. If the transmission is not received successfully, as indicated by the CRC check, then the partner notifies the mobile about it, and the mobile transmits the rest of the coded bits (symbols) itself, instead of relying on cooperation from its partner. The codes are designed to provide the best possible performance for this case as well [7]. Note that, as far as the destination is concerned, it does not matter whether the transmission comes from the mobile or its partner, since the channel is estimated periodically and the decoding algorithm remains unchanged. The only difference is in the achieved performance, i.e., diversity level.
A.2 Adaptive Protocol

Instead of the fixed protocol, it may be more advantageous to use an adaptive protocol. Since the initial transmission from the mobile will be received by the destination, it can decode the data and perform a CRC check to find out whether the transmission was received successfully. If so, it can notify the mobiles, hence there would be no need for cooperation to take place. In this case the mobile could continue its transmission. This type of protocol is especially appealing due to the fact that the mobiles use space-time codes which already provide error protection and diversity gains, hence it may not be necessary to cooperate all the time. Another advantage of the adaptive protocol is an increase in the data rates.

III. Performance Analysis

In this section, we provide an analysis of the cooperative space-time coding approach for wireless networks. We focus on the decode and forward protocol. We assume that mobile $M_1$ has $n_1$ antennas, mobile $M_2$ has $n_2$ antennas, and the destination has $m$ receive antennas. Without loss of generality, we study the diversity level achieved by user $M_1$ through cooperative space-time coding for various inter-user channel qualities. Similar analysis can also be performed for user $M_2$.

Let $P^C_f$ denote the frame error probability of the space-time channel code when user-cooperation is employed. This probability may be obtained as

$$P^C_f = \left(1 - P^{in}_f\right)P^{BF}_f + P^{in}_f P^{QS}_f$$

where $P^{in}_f$ denotes the frame error probability of the inter-user channel, $P^{BF}_f$ denotes the frame error probability over the block fading channel when cooperation takes place, and $P^{QS}_f$ denotes the frame error probability over the quasi-static fading channel in the non-cooperative case. We can find an upper bound on $P^C_f$ as

$$P^C_f \leq P^{BF}_f + P^{in}_f P^{QS}_f.$$ 

Let $E_{si}/N_0$ denote the received signal-to-noise ratio at the destination corresponding to the transmission from user $M_1$. Similarly, let $E_{s2}/N_0$ denote the received signal-to-noise ratio at the destination corresponding to the transmission from user $M_2$ and $E_{s3}/N_0$ denote the received signal-to-noise ratio at user $M_2$ corresponding to the transmission from user $M_1$.

Note that in the block fading model resulting from user cooperation, each block has a different received signal-to-noise ratio. However, the pairwise error probability can be derived in a form similar to [10]. Hence, utilizing the pairwise error probability expressions developed in [10] for the block Rayleigh fading channel and the union upper bound on the frame error probability, we can find upper bounds for $P^{BF}_f$, $P^{QS}_f$ and $P^{in}_f$.

For the upper bound in the case of block fading, we obtain [10]

$$P^{BF}_f \leq \sum_{c} \sum_{e \neq c} \prod_{b=1}^{B} \left( \frac{\mu_b E_{sc}}{4N_0} \right)^{-d_{c,m}}$$

where $d_b = \text{rank}(c[e_b] - e[b]) = \text{rank}(Z[b])$, and $Z$ denotes the code symbol difference matrix, $Z = c - e$. Also, $\mu = (\prod_{b=1}^{B} \lambda_1[b] \lambda_2[b] \cdots \lambda_{d_c}[b])^{1/d_c}$, and $\lambda_1[b], \lambda_2[b], \cdots, \lambda_{d_c}[b]$ are the nonzero eigenvalues of $Z[b] Z[b]^H$.

For the upper bound in the quasi-static fading channel case, $B = 1$, hence

$$P^{QS}_f \leq \sum_{c} \sum_{e \neq c} \left( \prod_{i=1}^{r} \lambda_i \right)^{-m} \left( \frac{E_{sc}}{4N_0} \right)^{-r m}$$

where $r$ is the rank of the code symbol difference matrix, $Z$, between the two entire codewords, and $\lambda_1, \lambda_2, \cdots, \lambda_r$ are the nonzero eigenvalues of $Z Z^H$.

Note that the inter-user channel is also quasi-static, hence

$$P^{in}_f \leq \sum_{c} \sum_{e \neq c} \left( \prod_{i=1}^{d_1} \gamma_i \right)^{-n_1} \left( \frac{E_{sc}}{4N_0} \right)^{-n_1 m}.$$ 

where $d_1$ is the rank of the code symbol difference matrix, $Z[1]$, between the two parts of the codewords that are used in the inter-user channel and $\gamma_1, \gamma_2, \cdots, \gamma_{d_1}$ are the nonzero eigenvalues of $Z[1] Z[1]^H$. Note that the number of receive antennas in the inter-user channel is $n_2$.

Therefore when mobile $M_1$ transmits in cooperation with mobile $M_2$, the upper bound on the frame error probability, $P^{C}_f$, for $M_1$ is

$$P^{C}_f \leq \left( \sum_{c} \sum_{e \neq c} \prod_{b=1}^{B} \left( \frac{\mu_b E_{sc}}{4N_0} \right)^{-d_{c,m}} \right)$$

$$+ \left( \sum_{c} \sum_{e \neq c} \left( \prod_{i=1}^{d_1} \gamma_i \right)^{-n_1} \left( \frac{E_{sc}}{4N_0} \right)^{-n_1 m} \right)$$

$$+ \left( \sum_{c} \sum_{e \neq c} \left( \prod_{i=1}^{r} \lambda_i \right)^{-m} \left( \frac{E_{sc}}{4N_0} \right)^{-r m} \right).$$

A. Asymptotic Analysis

At this point, we focus on two different cases. In the first case we assume that we have a good inter-user channel, and we study the performance at high signal-to-noise ratios for all channels. In the second case we will consider the situation when the inter-user channel is of poor quality. Our goal is to determine the achievable diversity in both cases.

A.1 Good Inter-User Channel

In the case when the inter-user channel is very good, i.e., it has a very high signal-to-noise ratio, $P^{in}_f$ is small and we simply have $P^{C}_f \approx P^{BF}_f$. Under the assumption that
we use full diversity codes and that \( E_{s_1} \approx E_{s_2} \approx E_s \), we obtain

\[
P_f^C \approx \left( \frac{E_s}{4N_0} \right)^{- (n_1 + n_2)m} \left\{ \sum_{c} \sum_{e \in c} \left( \frac{1}{\mu_1} \right)^{n_1m} \left( \frac{1}{\mu_2} \right)^{n_2m} \right\}.
\]

For simplicity, at high signal-to-noise ratios, the above bracketed expression may be approximated with the most dominant term in the summation, which may be denoted by, say \( K \), yielding the following approximation at high signal-to-noise ratios

\[
P_f^C \approx K \left( \frac{E_s}{4N_0} \right)^{- (n_1 + n_2)m}.
\]

This demonstrates that cooperative space–time coding can indeed achieve the full user cooperation diversity, as clearly indicated by the exponent of the signal-to-noise ratio.

Next, we focus on the case when \( E_{s_1} \approx E_{s_2} \approx E_{s_{\infty}} = E_s \) and all channels, including the inter–user channel, have similar quality. This assumption simplifies the diversity analysis and is quite reasonable at high signal-to-noise ratios in all channels. In this case \( P_f^C \) can be approximately upper bounded by

\[
P_f^C \leq E^{-(n_1m + n_2)m} \left( \sum_c \sum_{e \in c} \sum_{i=1}^2 \left( \frac{1}{\mu_i} \right)^{n_i m} \right)
+ \left( \frac{E_s}{4N_0} \right)^{- (n_1 + n_2)m} \left( \sum_c \sum_{e \notin c} \sum_{i=1}^2 \left( \frac{1}{\gamma_i} \right)^{n_i m} \right)
+ \left( \sum_c \sum_{e \notin c} \left( \frac{1}{\gamma_i} \right)^{n_i m} \right).\]

Let \( k = \min \{n_1, m\} \), which at high signal-to-noise ratios, leads to the following approximation

\[
P_f^C \approx K \left( \frac{E_s}{4N_0} \right)^{- n_1 m + n_2 k}.
\]

This indicates that the diversity order achieved through cooperative space–time coding depends on \( k = \min \{n_1, m\} \), as indicated by the exponent of the signal-to-noise ratio.

A.2 Poor Inter–User Channel

In this case, we assume that the inter–user channel has poor quality. Hence, the upper bound on \( P_f^C \) is dominated by the term \( P_{f_{\text{int}}}^C P_{f_{\text{int}}}^2 \). We obtain

\[
P_f^C \leq \left( \sum_c \sum_{e \notin c} \left( \frac{E_{s_{\infty}}}{4N_0} \right)^{- d_{e \notin c} m} \left( \frac{1}{\gamma_i} \right)^{n_i m} \right)
+ \left( \sum_c \sum_{e \notin c} \left( \frac{E_{s_{\infty}}}{4N_0} \right)^{- d_{e \notin c} m} \left( \frac{1}{\gamma_i} \right)^{n_i m} \right).
\]

As the inter–user channel quality is low, we can assume that the inter–user channel signal-to-noise ratio is approximately constant for all transmit powers of interest, i.e.,

\[
\left( \frac{E_{s_{\infty}}}{4N_0} \right) \approx C_{\text{in}}. \]

We have

\[
P_f^C \leq \left( \sum_c \sum_{e \notin c} \frac{1}{C_{\text{in}}} \left( \prod_{i=1}^n \frac{1}{\gamma_i} \right)^{n_i m} \right)
+ \left( \sum_c \sum_{e \notin c} \left( \frac{E_{s_{\infty}}}{4N_0} \right)^{- d_{e \notin c} m} \left( \frac{1}{\gamma_i} \right)^{n_i m} \right).
\]

At high signal-to-noise ratios in the user–destination channel, we obtain the following approximation for \( P_f^C \),

\[
P_f^C \approx \frac{1}{C_{\text{in}}} \left( \frac{E_s}{4N_0} \right)^{- m} \min_{c,e} \left\{ \left( \prod_{i=1}^n \frac{1}{\gamma_i} \right)^{n_i m} \right\}
\]

where \( \min_{c,e} \left\{ \left( \prod_{i=1}^n \frac{1}{\gamma_i} \right)^{n_i m} \right\} \) denotes the minimum product of the codes eigenvalues which dominates the performance at high signal-to-noise ratios.

Hence, for poor inter–user channel quality, the diversity is limited to the diversity of the non–cooperative quasi–static fading channel. It is maximum for \( r = n_1 \). Despite the limited diversity there is still some coding gain with respect to the non–cooperative space–time coding case, as indicated by the eigenvalue product. Hence, we will still observe performance improvements with respect to the non–cooperative coding case.

B. Evaluation of the Union Bound

Unfortunately, in the case of slow, quasi–static or block fading, the union bound evaluated in a straightforward manner would be loose. This is due to the fact that there are no dominant error events [11].

We can tighten the bound by performing expurgation of the union bound [13]. Namely, it can be shown that all erroneous sequences which differ from the correct sequence in more than one simple error event may be dropped from the bound. However, it has been observed in [14], [15], that this technique alone is not sufficient to obtain a tight upper bound on the performance of space–time trellis codes. Therefore, we combine the technique proposed by Mokrasi and Leib [11], with the idea in [13], that is, we limit the conditional union upper bound on the error probability before averaging over the fading distribution. In the standard union bound approach in order to obtain the bound on the probability of frame error, we need to perform averaging over the fading statistics

\[
P_f \leq \int_{\alpha} \frac{1}{|S|} \sum_c \sum_{e \notin c} P(c \rightarrow e|\alpha)f(\alpha)\,d\alpha
\]

where \( |S| \) denotes the number of codewords in the space–time code. However, as this approach results in loose
bounds, in order to evaluate the probability of frame error we have

\[ P_f \leq \int_{\alpha} \min \left[ 1, \frac{1}{|\mathcal{S}|} \sum_{c \in \mathcal{C}} \sum_{e \neq c} P(c \rightarrow e | \alpha) \right] f(\alpha) d\alpha \]

where \( \alpha \) is the vector of channel gains, \( f(\alpha) \) is the joint probability density function of \( \alpha \) and \( P(c \rightarrow e | \alpha) \) denotes the conditional pairwise error probability given that the channel gains are \( \alpha \). We can write,

\[ P_f \leq \int_{\alpha} \min \left[ 1, \frac{1}{|\mathcal{S}|} \sum_{c \in \mathcal{C}} \sum_{e \neq c} Q \left( \frac{d^2(c, e) E_n}{2N_0} \right) \right] f(\alpha) d\alpha. \]

The distance term \( d^2(c, e) \) depends on the number of transmit and receive antennas. For example, for \( n_1 = n_2 = 2 \) transmit antennas at each mobile and \( m \) receive antennas at the destination, we have

\[
d^2(c, e) = \sum_{b=1}^{B} \sum_{l=1}^{L} \sum_{j=1}^{m} \alpha_i^b \left( c_{i,l} - e_{i,l} \right)^2
\]

\[
= \sum_{b=1}^{B} \left( \sum_{j=1}^{m} \alpha_i^b \right)^2 \sum_{l=1}^{L} \left( c_{i,l} - e_{i,l} \right)^2
\]

\[
+ \sum_{b=1}^{B} \left( \sum_{j=1}^{m} \alpha_i^b \right)^2 \sum_{l=1}^{L} \left( c_{i,l} - e_{i,l} \right)^2
\]

\[
+ 2 \text{Re} \left\{ \sum_{b=1}^{B} \sum_{j=1}^{m} \alpha_i^b \sum_{l=1}^{L} \left( c_{i,l} - e_{i,l} \right)^2 \right\}
\]

In order to compute the summation we need to keep track of the real variables \( A_{t,b}^1, A_{t,b}^2, B_{t,b}^1, B_{t,b}^2 \), where \( B_{t,b} = B_{t,b}^1 + jB_{t,b}^2 \) over all codeword pairs, and count the multiplicities of all the possibilities over the entire code [15].

Due to the minimization in the argument of the integral, the order of integration and summation cannot be interchanged and we have to perform numerical integration. However, as also demonstrated in [11] the increased numerical complexity becomes less of a problem due to the growing computing power. There are efficient routines [11] for numerical evaluation of integrals. Another possibility is the use of Monte Carlo techniques [17]. Monte Carlo methods are appealing, since they represent a simple method of performing the integration, by averaging over many realizations of the fading vector \( \alpha \).

Using the above bounds on the frame error probability, we can compute \( P_f^{m}, P_f^{QS} \) and \( P_f^{BF} \), which can then be used to determine the bound on the cooperative frame error probability. We will discuss the numerical results in the next section.

IV. NUMERICAL EXAMPLES

We present examples with the 4-PSK, 4 state space–time trellis code proposed in [18]. The code size is 520 symbols. The code achieves full spatial diversity over the quasi–static fading channel with two transmit antennas. In addition, by proper multiplexing of the coded output symbols and their transmission over the two independently faded blocks, this code can also be formatted to achieve the full diversity over the overall block fading channel, in the case of user–cooperation [8]. We consider the union bound on the frame error probability, with expurgation and limiting before averaging, in order to present examples for various qualities of the inter-user channel. In all examples we assume that the users experience quasi–static Rayleigh fading and the user–destination channels are of similar quality.

Figure 4 presents the upper bound on the frame error probability for the 4-PSK, 4 state space–time trellis code, for good inter-user channel quality. We focus on the case when the inter-user channel is perfect. We also present the result when the inter-user channel frame error probability is 0.1. We observe that when the inter-user channel is perfect, the code achieves the full diversity available with user–cooperation. Similarly, there is only a small loss in performance when the inter-user channel frame error probability is 0.1 as is also predicted by the bound.

![Figure 4: Upper bound on the frame error probability for different inter-user channel qualities, two transmit antennas and two receive antennas.](image)

Figure 5 presents the upper bound on the frame error probability for the 4-PSK, 4 state space–time trellis code, for poor inter-user channel quality. We consider the case when the inter-user channel frame error probability is 0.5. For comparison we also present the bound in the non–cooperative case, when the mobile experiences quasi–static fading. We observe that for poor inter-user channel quality the diversity is limited by the diversity of the
non-cooperative channel. This is clearly captured by the bound. Note that the bound accurately predicts the coding gain that user-cooperation provides even with such limited cooperation.

V. CONCLUSIONS

In this paper we analyzed the performance of cooperative space-time coding with the decode and forward protocol. We performed an asymptotic analysis in order to determine the achieved diversity order through cooperative coding for various inter-user channel qualities. We observed that for good inter-user channel quality, full diversity could be achieved through user-cooperation. The diversity gains decrease with the decrease in the inter-user channel quality, but there are still coding gains. In addition, we derived a tight bound on the frame error probability of the cooperative space-time trellis coded system. We demonstrated the usefulness of the bound through numerical examples.

REFERENCES


Fig. 5. Upper bound on the frame error probability for different inter-user channel qualities, two transmit antennas and two receive antennas.