The relation of description rate and investment growth rate

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Abstract — We have shown that if one invests in the outcome of a random variable \( X \), where investment consists of gambling at any odds, then every bit of description of \( X \) increases the doubling rate by one bit. However, if the provider of the information has access only to \( V \), a random variable jointly distributed with \( X \), then this maximal efficiency is not generally possible. We find the increase \( \Delta(R) \) in doubling rate for a description of \( V \) at rate \( R \) for the jointly Gaussian and jointly binary cases. We investigate the extension to multivariate Gaussian random variables. We prove a general result for the derivative of \( \Delta(R) \) at \( R = 0 \).

We then consider the problem in which there are \( k \) separate encoders and each observes a random variable \( V_i \) correlated with \( X \). We find how efficiently these encoders, without cooperation, help the investor who is interested in \( X \).

SUMMARY

Suppose one gambles on the outcome of a random variable \( X \). The investor distributes his wealth according to \( b(x) \) and the investment pays odds of \( o(x) \) for one. Also suppose that the description of another random variable \( V \), which has a known joint distribution with \( X \), at the rate of \( R \) bits is allowed. Let \( \Delta(R) \) be the maximum increase in the doubling rate from no description to a description of rate \( R \). It can be seen that \( \Delta(R) \) is a concave and nondecreasing function of \( R \). We can show [2] that

\[
\Delta(R) = \max_{p(\{V, x\}) : I(V; X) \leq R, V \to V \to X} I(V; X).
\]

We define initial efficiency as the derivative of \( \Delta(R) \) at the origin. Initial efficiency is the maximum possible increase in \( \Delta(R) \) per bit of description. For \( V = X \), \( \Delta(R) = R \); hence the efficiency is 1. However, for a general \( V \), the efficiency is generally less than 1. We find \( \Delta(R) \) and examine the efficiency of the jointly binary and Gaussian cases.

Theorem 1 Suppose \( V \) and \( X \) are both Bernoulli(\( \frac{1}{2} \)) random variables associated by a binary symmetric channel with crossover probability \( p \). The \( \Delta(R) \) curve is given by

\[
\Delta(R) = (1 - h(\alpha), 1 - h(\alpha \ast p))
\]

where \( 0 \leq \alpha \leq 1 \), \( h \) is the binary entropy function and \( \ast \) is the cascade operation.

We use a lemma by Wyner and Ziv, known as ‘Mrs. Gerber’s Lemma’ [4] to prove the optimality of the descriptions in the above theorem. The initial efficiency can be calculated as \((1 - 2p)^2\).

Theorem 2 Suppose \( X \) and \( V \) are jointly Gaussian with correlation \( \rho \). Then

\[
\Delta(R) = \frac{1}{2} \log \left( 1 - \rho^2 (1 - 2R) \right).
\]

The proof of optimality in the Gaussian problem requires a lemma by Bergmans, which is a conditional version of the entropy power inequality [1]. We note that the initial efficiency is \( \rho^2 \).

A natural generalization of this theorem is to multivariate Gaussian. Suppose \( V = N(0, K_{11}), Z = N(0, K_{22}), V_1 \) and \( Z \) are independent and \( X = V_1 + Z \). By changing the coordinate system \( \beta \), we can obtain diagonal covariance matrices and hence transform the problem to one on parallel subchannels with a total rate constraint. The solution is given by water-filling in the entropy domain. We distribute the total rate so that the derivative of \( \Delta(R) \) with respect to \( R \) at the operating point is the same for all the subchannels used.

We note that in all the problems examined, the initial efficiency is related to the correlation between \( V \) and \( X \). We define the maximal correlation between \( V \) and \( X \) as the supremum of \( EF(X)g(V) \), where the supremum is over all functions \( f \) and \( g \) such that \( EF(X) = Eg(V) = 0 \) and \( Ef(X) = Eg^2(V) = 1 \). Maximal correlation depends only on the joint distribution of \( V \) and \( X \) and is independent of the actual labeling. Conditions under which the maximal correlation can be attained have been investigated by Renyi [3]. Our next theorem examines the relationship between the initial efficiency and maximal correlation.

Theorem 3 Initial efficiency is equal to the square of the maximal correlation between \( V \) and \( X \).

Next we consider \( k \) separate senders. We are interested in the increase in the doubling rate, \( \Delta \), for gambling on \( X \) when sender \( i \) observes \( V_i \) correlated with \( X \) and the senders operate at respective rates \( R_1, \ldots, R_k \). We prove an achievable region for \( (R_1, \ldots, R_k, \Delta) \), and show that a Slepian-Wolf type of rate region is achievable for this investment problem.

REFERENCES


