Solving the Optimal PWM Problem for Single-Phase Inverters

Dariusz Czarkowski, *Member, IEEE*, David V. Chudnovsky, *Member, IEEE*, Gregory V. Chudnovsky, and Ivan W. Selesnick, *Member, IEEE*

***Abstract*—In this paper, the basic algebraic properties of the optimal PWM problem for single-phase inverters are revealed. Specifically, it is shown that the nonlinear design equations given by the standard mathematical formulation of the problem can be reformulated, and that the sought solution can be found by computing the roots of a single univariate polynomial $P(x)$, for which algorithms are readily available. Moreover, it is shown that the polynomials $P(x)$ associated with the optimal PWM problem are orthogonal and can therefore be obtained via simple recursions. The reformulation draws upon the Newton identities, Padé approximation theory, and properties of symmetric functions. As a result, fast $O(n \log^2 n)$ algorithms are derived that provide the exact solution to the optimal PWM problem. For the PWM harmonic elimination problem, explicit formulas are derived that further simplify the algorithm.**

***Index Terms*—Harmonic elimination, Newton identities, orthogonal polynomials, Padé approximation, pulsewidth modulation (PWM), single-phase inverters, symmetric functions.**

I. Introduction

THE PROBLEM of the optimal design of pulsewidth modulated (PWM) waveforms for single-phase inverters [1], [2] is examined in this paper. PWM signals are used in power electronics, motor control and solid-state electric energy conversion [2], [3]. The best voltage signal for these purposes is one with a periodic time variation in which amplitudes of selected nonfundamental components of the signal have been controlled to increase efficiency and reduce damaging vibrations. How can the waveform be designed to most effectively accomplish this? This paper describes a contribution to the theory and practice of optimal PWM waveforms that addresses this question.

A PWM waveform consists of a series of positive and negative pulses of constant amplitude but with variable switching instances as depicted in Fig. 1 (as in a power electronic PWM full-bridge inverter). A typical goal is to generate a train of pulses such that the fundamental component of the resulting waveform has a specified frequency and amplitude (e.g., for a constant V/f speed control of an induction motor). Some of

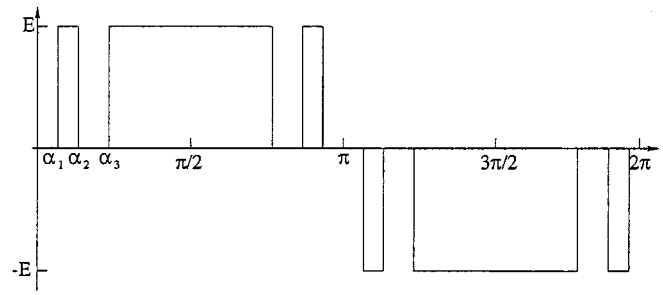

Fig. 1. A three-level PWM waveform.

the proposed methods for PWM waveform design are: modulating-function techniques, space-vector techniques, and feedback methods [2]. These methods suffer, however, from high residual harmonics that are difficult to control and from limitations in their applicability. A method that theoretically offers the highest quality of the output waveform is the so-called programmed or optimal PWM [4]. Owing to the symmetries in the PWM waveform of Fig. 1, only the odd harmonics exist. Assuming that the PWM waveform is chopped n times per half a cycle, the Fourier coefficients of odd harmonics are given by

$$V_k = \frac{4E}{k\pi}\left[\cos k\alpha_1 - \cos k\alpha_2 + \cdots + (-1)^{j-1}\cos k\alpha_j + \cdots + (-1)^{n-1}\cos k\alpha_n\right] \quad (1)$$

where $k = 1, 3, 5, \ldots$ and E is the amplitude of the square wave. Amplitudes of any n harmonics can be set by solving a system of n nonlinear equations obtained from setting (1) equal to prespecified values. In the harmonic elimination programmed PWM method, the fundamental component is set to a required amplitude and $n-1$ low-order harmonics are set to zero. This is the most common approach in electric drives since low-order harmonics are the most detrimental to motor performance. In other applications, like active harmonic filters or control of electromechanical systems, harmonics are set to nonzero values. This task of designing a PWM waveform, the first n Fourier series coefficients of which match those of a desired waveform has been the subject of many papers [1], [5]–[37]. Often, the Newton iteration method [5] or an unconstrained optimization approach [38] are used to solve the system of nonlinear equations (1). Those methods are computationally intensive for on-line calculations and the storage of off-line calculations leads to high memory requirements. Recent results from the research community [39], [40] show two approaches to real-time implementation of an approximate optimal PWM. One approach is to fine tune conventional PWM techniques like regular-sampled

Manuscript received December 13, 1999; revised January 20, 2001. This work was supported in part by the National Science Foundation under CAREER Grant CCR-987452. This paper was recommended by Associate Editor H. S. H. Chung.

D. Czarkowski and I. W. Selesnick are with the Department of Electrical and Computer Engineering, Polytechnic University, Brooklyn, NY 11201 USA (e-mail: dcz@pl.poly.edu; selesi@taco.poly.edu).

D. V. Chudnovsky and G. V. Chudnovsky are with the Department of Mathematics, Polytechnic University, Brooklyn, NY 11201 USA.

Publisher Item Identifier S 1057-7122(02)03137-9.

[39] or space-vector methods [41], [40] to approximate programmed PWM switching patterns. Another approach is to simplify the nonlinear harmonic elimination equations [26] in order to obtain real-time approximate solutions using modern digital signal processors.

This paper develops $O(n \log^2 n)$ algorithms for solving the PWM harmonic elimination problem without any approximations in the problem statement. Since many PWM applications allow for a computational time frame of a few milliseconds, the developed algorithms will allow for real time generation of switching patterns with n of the order of hundreds.

II. Converting the PWM Problem

For the scope of this paper, a PWM waveform is a 2π-periodic function $f(t)$ that is binary-valued for $0 \le t \le \pi/2$ and has the symmetries $f(t) = f(\pi - t)$ and $f(t) = -f(2\pi - t)$ as shown in Fig. 1. As such, $f(t)$ can be written with the Fourier series as

$$f(t) = \sum_{k=1}^{\infty} f_{2k-1} \cos(2k-1)t$$

with

$$f_k = \frac{2}{\pi} \int_0^{\pi/2} f(t) \cos(kt)\, dt.$$

The optimal PWM problem, as it is considered here, is the design of a PWM waveform $f(t)$ so that its first Fourier coefficients f_k are equal to prescribed values. For a PWM waveform, as shown in Fig. 1, we have

$$f_k = \frac{4E}{\pi k} \sum_{i=1}^{n} (-1)^{i-1} \cos k\alpha_i.$$

Therefore, the optimal PWM problem gives rise to the following design equations [4]:

$$\begin{aligned} \cos\alpha_1 - \cos\alpha_2 + \cos\alpha_3 - \cdots \cos\alpha_n &= h_1 \qquad (2)\\ \cos 3\alpha_1 - \cos 3\alpha_2 + \cos 3\alpha_3 - \cdots \cos 3\alpha_n &= h_3\\ &\vdots\\ \cos(2n-1)\alpha_1 - \cos(2n-1)\alpha_2 &\\ + \cos(2n-1)\alpha_3 - \cdots \cos(2n-1)\alpha_n &= h_{2n-1}. \qquad (3) \end{aligned}$$

Given the n values $h_k = k\pi f_k/(4E)$, we have n equations and n unknowns; we would like to find the n unknowns $\{\alpha_1, \ldots, \alpha_n\}$, with $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \pi/2$. We can first simplify the equations as is done for example in [22].

Let $\beta_i = \alpha_i$ for odd i, and let $\beta_i = \pi - \alpha_i$ for even i. Then

$$\begin{aligned} \cos\beta_1 + \cos\beta_2 + \cos\beta_3 + \cdots + \cos\beta_n &= h_1 \qquad (4)\\ \cos 3\beta_1 + \cos 3\beta_2 + \cos 3\beta_3 + \cdots + \cos 3\beta_n &= h_3\\ &\vdots\\ \cos(2n-1)\beta_1 + \cos(2n-1)\beta_2 &\\ + \cos(2n-1)\beta_3 + \cdots + \cos(2n-1)\beta_n &= h_{2n-1}. \qquad (5) \end{aligned}$$

Note that $\cos nt = T_n(\cos t)$ where T_n is the nth Chebyshev polynomial. Let $x_i = \cos\beta_i$, then

$$\begin{aligned} T_1(x_1) + T_1(x_2) + \cdots T_1(x_n) &= h_1\\ T_3(x_1) + T_3(x_2) + \cdots T_3(x_n) &= h_3\\ &\vdots\\ T_{2n-1}(x_1) + T_{2n-1}(x_2) + \cdots T_{2n-1}(x_n) &= h_{2n-1}. \end{aligned}$$

As the odd-indexed Chebyshev polynomials are odd polynomials, we can write

$$T_{2k-1}(x) = \sum_{m=1}^{k} c_{k,m} x^{2m-1}.$$

With this notation, the PWM equations become

$$\sum_{i=1}^{n} \sum_{m=1}^{k} c_{k,m} \cdot x_i^{2m-1} = h_{2k-1}, \qquad 1 \le k \le n$$

or

$$\sum_{m=1}^{k} c_{k,m} \cdot s_{2m-1} = h_{2k-1}, \qquad 1 \le k \le n \qquad (6)$$

where

$$s_m = \sum_{i=1}^{n} x_i^m$$

are the sums of powers of $\{x_i\}$. Equation (6) forms a set of n linear equations for s_{2m-1}, $1 \le m \le n$. Once the values s_{2m-1} are obtained by solving the linear system (6), one has the following problem. Given $\{s_1, s_3, \ldots, s_{2n-1}\}$, find the solution $\{x_1, x_2, \ldots, x_n\}$ to the following system of nonlinear equations

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= s_1 \qquad (7)\\ x_1^3 + x_2^3 + \cdots + x_n^3 &= s_3\\ &\vdots\\ x_1^{2n-1} + x_2^{2n-1} + \cdots + x_n^{2n-1} &= s_{2n-1}. \qquad (8) \end{aligned}$$

Once x_i are obtained, the original variables α_i can be found by letting $\beta_i = \arccos x_i$, $\alpha_i = \beta_i$ for odd i, and $\alpha_i = \pi - \beta_i$ for even i. Due to the symmetry with respect to x_i, any permutation of a solution set $\{x_i\}$ is also a solution set; likewise for β_i. Yet it is necessary to order β_i appropriately such that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \pi/2$. Note that for odd i, $0 < \alpha_i < \pi/2$ gives $0 < \beta_i < \pi/2$. For even i, $0 < \alpha_i < \pi/2$ gives $0 < \pi - \beta_i < \pi/2$ or $\pi/2 < \beta_i < \pi$. This indicates how to obtain α_i with the desired ordering from β_i: for those values of $\beta_i \in (0, \pi/2)$ let $\alpha_i = \beta_i$; and for those values of $\beta_i \in (\pi/2, \pi)$ let $\alpha_i = \pi - \beta_i$.

However, the design equations (7) and (8) are nonlinear, so obtaining the desired solution $\{x_i\}$ is not so straightforward. In the following sections, this nonlinear system of equations will be closely examined and in Section III, a systematic procedure is given to obtain the solutions.

A. Specialization to the Harmonic Elimination Problem

For the harmonic elimination PWM problem, which we will focus on below, the Fourier coefficients of the PWM waveform $f(t)$ should match the Fourier coefficients of a pure sine wave. That is, the values h_{2k-1} appearing in (2) and (3) are given by $h_1 = A$, and $h_{2i-1} = 0$ for $2 \le i \le n$. For this case, the values s_{2i-1} depend on A only and are given by

$$s_{2i-1} = \frac{A}{4^{i-1}}\binom{2i-1}{i-1}, \qquad 1 \le i \le n.$$

B. A Simpler Problem

It is useful to consider first the related, simpler problem where contiguous sums of powers are known. Given the values $s_1, \ldots, s_n$; find x_i satisfying the following equations:

$$x_1 + x_2 + \cdots + x_n = s_1 \tag{9}$$
$$x_1^2 + x_2^2 + \cdots + x_n^2 = s_2$$
$$\vdots$$
$$x_1^n + x_2^n + \cdots + x_n^n = s_n. \tag{10}$$

It turns out that $\{x_1, \ldots, x_n\}$ are the roots of the polynomial

$$P(x) = x^n + p_1 x^{n-1} + \cdots + p_n = 0 \tag{11}$$

where

$$\begin{aligned} s_1 + p_1 &= 0 \\ s_2 + p_1 s_1 + 2p_2 &= 0 \\ s_3 + p_1 s_2 + p_2 s_1 + 3p_3 &= 0 \end{aligned}$$

etc. The coefficients p_k can be found as

$$\begin{aligned} p_1 &= -s_1 \\ p_2 &= -\tfrac{1}{2}(s_2 + p_1 s_1) \\ p_3 &= -\tfrac{1}{3}(s_3 + p_1 s_2 + p_2 s_1) \\ &\vdots \\ p_k &= -\frac{1}{k}\left(s_k + p_1 s_{k-1} + \cdots + p_{k-1} s_1\right) \end{aligned} \tag{12}$$

etc. These are *Newton's Identities*. When contiguous values s_i are known one can easily obtain p_k through these simple recursive relations. Then x_i can be obtained by taking the roots of (11) for which numerical algorithms are readily available. (For $k > n$ these identities involve coefficients p_k beyond the highest power, which are implicitly defined as 0.) For the PWM problem outlined above, we know only the first n *odd* s_k, for $k = 1, 3, \ldots, 2n-1$. We do not know contiguous values s_i; therefore Newton's identities cannot be used. Our question is: Does any such simple procedure exist for the harmonic elimination problem for PWM, where one is given only the odd s_k, for $k = 1, 3, 5, \ldots, 2n-1$?

Note that (7)–(10) are symmetric functions of $\{x_i\}$. The problem of finding the set of n elements $\{x_i\}$ with given values of n *arbitrary* symmetric functions $\{s_j\}$ in x_i, $i = 1 \cdots n$, is in general a very complicated one because of nontrivial nature of relations between symmetric functions of high degrees. Only in special cases this problem can be reduced to a manageable and "exactly solvable" one. The classical cases of sum of powers (Newton), Wronski and elementary symmetric functions are the ones well described in the literature (see, e.g., [42] on representation of symmetric groups and Frobenius and Schur functions).

In Section III, we present a procedure to determine the polynomial $P(x)$ for the case where one is given the "sums of odd powers," as it appears in the optimal PWM problem.

III. Generalizing Newton's Identity

To develop a procedure to solve the optimal PWM problem, it will be useful to examine more closely Newton's identities. The simple derivation of Newton's identities will serve as our starting point. First write the polynomial $P(x)$ as

$$P(x) = \prod_{i=1}^{n}(x - x_i),$$

then the logarithmic derivative is given by

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n}\frac{1}{x - x_i}.$$

Expanding each term in the sum, one gets

$$\begin{aligned} \frac{P'(x)}{P(x)} &= \sum_{i=1}^{n}\sum_{m=0}^{\infty}\frac{x_i^m}{x^{m+1}} \\ &= \sum_{m=0}^{\infty}\frac{s_m}{x^{m+1}} \end{aligned} \tag{13}$$

where

$$s_m = \sum_{i=1}^{n} x_i^m$$

are the sums of the root powers, with $s_0 = n$.

Integrating (13) gives

$$\begin{aligned} \frac{P'(x)}{P(x)} &= \sum_{m=0}^{\infty}\frac{s_m}{x^{m+1}} = \frac{n}{x} + \sum_{m=1}^{\infty}\frac{s_m}{x^{m+1}} \\ \int \frac{P'(x)}{P(x)}\,dx &= \int \frac{n}{x}\,dx + \int \sum_{m=1}^{\infty}\frac{s_m}{x^{m+1}}\,dx \\ \ln P(x) &= n\ln x - \sum_{m=1}^{\infty}\frac{s_m}{m x^m}. \end{aligned}$$

Raising e to the power of $\ln P(x)$ and using the last equation gives

$$P(x) = x^n \exp\left(-\sum_{m=1}^{\infty}\frac{s_m}{m x^m}\right). \tag{14}$$

In order to generalize Newton's procedure, we will obtain an expression similar to (14), but having only odd s_i. To this end, note that

$$P(-x) = (-1)^n x^n \exp\left(-\sum_{m=1}^{\infty}\frac{s_m(-1)^m}{m x^m}\right)$$

and that

$$\frac{P(x)}{P(-x)} = (-1)^n \exp\left(-\sum_{m=1}^{\infty} \frac{s_m}{mx^m}\,(1-(-1)^m)\right)$$

or

$$\frac{P(x)}{P(-x)} = (-1)^n \exp\left(-2\sum_{m=1,\,m\,\mathrm{odd}}^{\infty} \frac{s_m}{mx^m}\right).$$

Then

$$P(x) = (-1)^n P(-x) G(1/x)$$

where

$$G(x) := e^{V(x)}$$
$$V(x) := -2\left(s_1 x + \frac{s_3}{3}\,x^3 + \frac{s_5}{5}\,x^5 + \cdots\right).$$

With the notation

$$\tilde{P}(x) = (-1)^n P(-x) \tag{15}$$

one has

$$P(x) = \tilde{P}(x) G(1/x). \tag{16}$$

$\tilde{P}(x)$ is the monic polynomial related to $P(x)$ by negating the roots of $P(x)$.

Equation (16) is the counter-part to (14). Likewise, by setting like powers of x equal, we can obtain equations that relate p_k and s_i. However, in order to do this, we need to expand $G(x) = e^{V(x)}$ into a power series of x. Such a power series for $e^{V(x)}$ can be obtained using the following algorithm, described in [43, Sec. 4.7]. Let

$$V(x) = \sum_{i=0}^{\infty} v_i x^i$$

and

$$G(x) = e^{V(x)} = \sum_{i=0}^{\infty} g_i x^i.$$

If v_i for $0 \le i \le m$ are known, then the values of g_i for $0 \le i \le m$ are given by

$$g_0 = e^{v_0} \tag{17}$$

$$g_i = \frac{1}{i}\sum_{k=1}^{i} k v_k g_{i-k} \tag{18}$$

for $1 \le i \le m$.

When the first n odd values of s_i are known, then v_i are known for $0 \le i \le 2n$. [$v_{2i} = 0$ for $0 \le i \le n$ and $v_{2i-1} = -2s_{2i-1}/(2i-1)$ for $1 \le i \le n$.] Therefore, using the relations (17) and (18), we obtain g_i for $0 \le i \le 2n$, and consequently, we can write out Equation (16), matching like powers of x, to obtain linear equations from which p_k can be obtained. For example, we write out the expressions for $n = 3$. That is, we are given s_1, s_3, and s_5, and our goal is to find the corresponding monic third degree polynomial $P(x)$,

$$P(x) = x^3 + p_1 x^2 + p_2 x + p_3.$$

$V(x)$ is given by

$$V(x) = -2s_1 x - \tfrac{2}{3}s_3 x^3 - \tfrac{2}{5}s_5 x^5 + ?x^7 + ?x^9 + \cdots.$$

The ? indicates unknown coefficients. Then, using (17) and (18), $G(x)$ can be determined up to and including g_6

$$G(x) = 1 + g_1 x + \cdots + g_6 x^6 + ?x^7 + ?x^9 + \cdots.$$

Writing out (16) gives

$$\begin{aligned}&(x^3 + p_1 x^2 + p_2 x + p_3)\\ &\quad = -(-x^3 + p_1 x^2 - p_2 x + p_3)(1 + g_1/x + g_2/x^2 + \cdots).\end{aligned}$$

Matching like powers of x gives

$$\begin{array}{lll} x^3 & \rightarrow & 1 = 1\\ x^2 & \rightarrow & p_1 = -p_1 + g_1\\ x^1 & \rightarrow & p_2 = p_2 - g_1 p_1 + g_2\\ x^0 & \rightarrow & p_3 = -p_3 + g_1 p_2 - g_2 p_1 + g_3\\ 1/x & \rightarrow & 0 = -g_1 p_3 + g_2 p_2 - g_3 p_1 + g_4\\ 1/x^2 & \rightarrow & 0 = -g_2 p_3 + g_3 p_2 - g_4 p_1 + g_5\\ 1/x^3 & \rightarrow & 0 = -g_3 p_3 + g_4 p_2 - g_5 p_1 + g_6. \end{array}$$

Matching further like powers involves coefficients g_7 and higher, which are unknown. Note that the last three equations, corresponding to $1/x$, $1/x^2$, and $1/x^3$, can be written as the following linear system of equations:

$$\begin{bmatrix} g_3 & -g_2 & g_1\\ g_4 & -g_3 & g_2\\ g_5 & -g_4 & g_3 \end{bmatrix}\begin{bmatrix} p_1\\ p_2\\ p_3\end{bmatrix} = \begin{bmatrix} g_4\\ g_5\\ g_6\end{bmatrix}.$$

This system is almost Toeplitz—only the signs of the odd columns must be negated. The true Toeplitz system

$$\begin{bmatrix} g_3 & g_2 & g_1\\ g_4 & g_3 & g_2\\ g_5 & g_4 & g_3 \end{bmatrix}\begin{bmatrix} \tilde{p}_1\\ \tilde{p}_2\\ \tilde{p}_3\end{bmatrix} = -\begin{bmatrix} g_4\\ g_5\\ g_6\end{bmatrix}$$

where $p_1 = -\tilde{p}_1$, $p_2 = \tilde{p}_2$, $p_3 = -\tilde{p}_3$ gives the solution for p_i. Note that $\tilde{p}_i$ are the coefficients of $\tilde{P}(x)$ defined in (15). As this system is Toeplitz (the diagonals of the system matrix are constant), the Levinson algorithm can be used to solve it efficiently [$O(n^2)$ instead of $O(n^3)$] [44].

The same procedure for general n gives rise to a Toeplitz system and comprises a generalization of Newton's identities to the case where only the first n *odd* values s_i are known. The $O(n^2)$ procedure for general n follows likewise.

Algorithm ODDSOL: Given the sums of odd powers s_{2i-1} for $1 \le i \le n$, the polynomial $P(x)$ can be obtained as follows.

1) Set $v_{2i} = 0$ for $0 \le i \le n$. Set $v_{2i-1} = -2s_{2i-1}/(2i-1)$ for $1 \le i \le n$.
2) Compute g_i for $0 \le i \le 2n$ using Equations (17) and (18).
3) Solve the Toeplitz system for $\tilde{p}_i$

$$\begin{bmatrix} g_n & \cdots & g_1\\ \vdots & \ddots & \vdots\\ g_{2n-1} & \cdots & g_n \end{bmatrix}\begin{bmatrix} \tilde{p}_1\\ \vdots\\ \tilde{p}_n\end{bmatrix} = -\begin{bmatrix} g_{n+1}\\ \vdots\\ g_{2n}\end{bmatrix} \tag{19}$$

and set $p_i = (-1)^i \tilde{p}_i$.

Algorithm HESOL: Given A and n, the steps to solve the harmonic elimination problem (2) and (3), are summarized as follows.

TABLE I
PARAMETER VALUES COMPUTED IN ALGORITHMS ODDSOL AND HESOL FOR THE PWM HARMONIC-ELIMINATION PROBLEM WITH $n = 4$ AND $A = 0.6$

$g_0 = +1.00000000000000$	$p_0 = +1.00000000000000$	$\beta_1 = 0.76848515715652\,\pi$
$g_1 = -1.20000000000000$	$p_1 = -0.60000000000000$	$\beta_2 = 0.52807085706244\,\pi$
$g_2 = +0.72000000000000$	$p_2 = -0.64755955745780$	$\beta_3 = 0.31666400048098\,\pi$
$g_3 = -0.58800000000000$	$p_3 = +0.31053573447468$	$\beta_4 = 0.15043709981329\,\pi$
$g_4 = +0.44640000000000$	$p_4 = +0.03190274246951$	$\alpha_1 = 0.15043709981329\,\pi$
$g_5 = -0.38673600000000$	$x_1 = -0.74695539133959$	$\alpha_2 = 0.23151484284348\,\pi$
$g_6 = +0.31554720000000$	$x_2 = -0.08807293772494$	$\alpha_3 = 0.31666400048098\,\pi$
$g_7 = -0.28238094857143$	$x_3 = +0.54464605975877$	$\alpha_4 = 0.47192914293756\,\pi$
$g_8 = +0.23942744228571$	$x_4 = +0.89038226930576$	

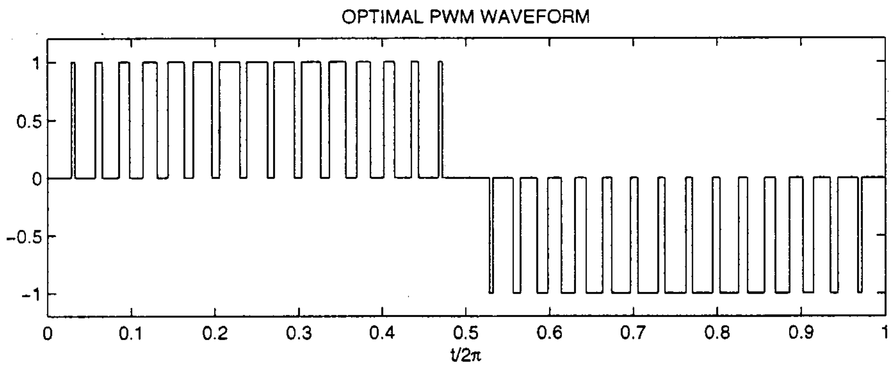

Fig. 2. The PWM harmonic-elimination waveform obtained, with $A = 0.6$ and $n = 15$.

1) Set $s_{2i-1} = (A/4^{i-1})\binom{2i-1}{i-1}$ for $i = 1, \ldots, n$.
2) From s_{2i-1}, $i = 1, \ldots, n$, obtain the degree n polynomial $P(x)$ for which the roots x_i satisfy the equations (7) and (8). (Use algorithm ODDSOL.) Find the roots x_i of $P(x)$.
3) Set $\beta_i = \arccos x_i$, $i = 1, \ldots, n$, with $\beta_i \in (0, \pi)$. For $\beta_i \in (0, \pi/2)$, set $\alpha_i = \beta_i$. For $\beta_i \in (\pi/2, \pi)$, set $\alpha_i = \pi - \beta_i$. Sort the angles α_i.

For the general optimal PWM problem, the values s_{2i-1} can be obtained by solving the linear system (6).

Example 1 (Harmonic Elimination): Let $A = 0.6$, $n = 4$. Then

$$s_1 = 0.6, \quad s_3 = 0.45, \quad s_5 = 0.375, \quad s_7 = 0.328\,125$$

and

$$(v_0, \ldots, v_8) = (0, -1.2, 0, -0.3, 0, -0.15, 0, -0.093\,75, 0).$$

Then, the parameters computed in algorithms ODDSOL and HESOL are shown in Table I. As $\beta_1, \beta_2 \in (\pi/2, \pi)$ and $\beta_3, \beta_4 \in (0, \pi/2)$ the α_i shown in the table are obtained by letting $\alpha_1 = \pi - \beta_1$, $\alpha_2 = \pi - \beta_2$, $\alpha_3 = \beta_3$, $\alpha_4 = \beta_4$, and then reordering α_i. It can be verified numerically that the harmonic elimination equations (2) and (3) are satisfied.

Example 2: Fig. 2 illustrates the PWM signal obtained with $A = 0.6$ and $n = 15$.

It should be noted that for the optimal PWM problem, the Toeplitz matrix in (19) becomes ill-conditioned as n grows larger. By using extended precision arithmetic, the range of n for which this basic algorithm is useful can be extended. However, in the following sections it will be shown that the solution can also be expressed as the solution to a Padé approximation problem, and consequently the polynomials $P_n(x)$ are orthogonal. Numerically stable algorithms using properties of orthogonal polynomials can therefore be developed and will be presented in future publications.

IV. A RECURRENCE FOR $P(x)$

The fast recursive algorithms for solving Toeplitz systems compute a solution using the solutions for smaller n. The notation $P_n(x)$ will be used to emphasize the dependence of $P(x)$ on n. Specifically, $P_n(x)$ denotes the monic degree-n polynomial associated with the harmonic elimination problem, with coefficients $p_{n,k}$

$$P_n(x) = x^n + p_{n,1}x^{n-1} + \cdots + p_{n,n}.$$

With this notation, a recurrence relation

$$P_{n+1}(x) = xP_n(x) + C_nP_{n-1}(x) \tag{20}$$

can be used to compute $P_n(x)$. For the harmonic elimination problem, the initial conditions can be taken to be $P_0(x) = 1$ and $P_1(x) = x - A$. The coefficients C_n in the recursion can be computed using the following formula:

$$C_n = -\frac{\sum_{k=0}^{n}(-1)^k g_{2n+1-k}p_{n,k}}{\sum_{k=0}^{n-1}(-1)^k g_{2n-1-k}p_{n-1,k}}. \tag{21}$$

This recurrence exists because of the Toeplitz structure, see [45] for further details. The coefficients $p_{n+1,k}$ are then determined recursively as [this implements (20)]

$$\begin{aligned} p_{n+1,k} &= p_{n,k}, & k &= 1\\ p_{n+1,k} &= p_{n,k} + C_n \cdot p_{n-1,k-2}, & k &= 2, \ldots, n\\ p_{n+1,k} &= C_n \cdot p_{n-1,k-2}, & k &= n+1. \end{aligned} \tag{22}$$

In order for this algorithm to work, one still needs the coefficients g_i in the expansion of $G(x)$. While g_i can be computed using (17) and (18), in Section V, a more direct recursion will be presented for computing the values g_i themselves.

V. A Recurrence for $G(x)$

The algorithms presented above for the PWM harmonic elimination problem do not fully utilize the fact that the parameters s_i depend only on the single parameter A. In this section we will show a relationship between the original (transcendental) formulation of the PWM problem [in terms of $\cos(2i-1)\beta$] and the converted (algebraic) formulation (in terms of x^{2i-1}). As a result, we obtain an expression for the function $G(1/x)$ that is important for developing a recursion for generating the coefficients g_i that are needed for computing C_n in the recursion $P_{n+1} = xP_n + C_nP_{n-1}$.

In this section, we show how the transcendental case can be explicitly expressed using the introduced notations of $G(1/x)$ and $P_n(x)$. The basic transformation is

$$\cos\beta_i = \tfrac{1}{2}(z_i + 1/z_i)$$

with $z_i = e^{j\beta_i}$. With the previous transformation $\cos\beta_i = x_i$ one has $x_i = 1/2(z_i + 1/z_i)$. Solving for z_i gives

$$z_i = x_i \pm \sqrt{x^2 - 1}.$$

Note that, as $\cos n\beta_i = 1/2(e^{jn\beta_i} + e^{-jn\beta_i})$, one gets

$$\cos n\beta_i = \tfrac{1}{2}(z_i^n + 1/z_i^n)$$

so that the equations (4) and (5) can now be written as

$$z_1 + 1/z_1 + z_2 + 1/z_2 + \cdots + z_n + 1/z_n = 2A \qquad (23)$$

$$\begin{aligned} &z_1^3 + 1/z_1^3 + z_2^3 + 1/z_2^3 + \cdots + z_n^3 + 1/z_n^3 = 0 \\ &\qquad\vdots \\ &z_1^{2n-1} + 1/z_1^{2n-1} + z_2^{2n-1} \\ &\quad + 1/z_2^{2n-1} + \cdots + z_n^{2n-1} + 1/z_n^{2n-1} = 0. \end{aligned} \qquad (24)$$

Therefore, if we define

$$P_a(z) = \prod_{i=1}^{n}(z - z_i), \qquad P_b(z) = \prod_{i=1}^{n}\left(z - \frac{1}{z_i}\right)$$

then, the product $P_a(z) \cdot P_b(z)$ has the roots $\{z_1, \ldots, z_n, 1/z_1, \ldots, 1/z_n\}$ and as such the key identity (16) can be applied directly to this problem—as it is of the "sums of odd powers" form. One gets

$$\frac{P_a(z)P_b(z)}{P_a(-z)P_b(-z)} = e^{-2\sum_{m=1,\, m\,\mathrm{odd}}^{\infty}(t_m/mz^m)}$$

where

$$\begin{aligned} t_m &= \sum_{i=1}^{m} z_i^m + 1/z_i^m \\ &= 2\sum_{i=1}^{m}\cos m\beta_i. \end{aligned}$$

From (23) and (24) we have for the harmonic elimination problem, that $t_1 = 2A$ and $t_{2i-1} = 0$ for $2 \le i \le n$ so that

$$\begin{aligned} \frac{P_a(z)P_b(z)}{P_a(-z)P_b(-z)} &= e^{-2(2A/z + O(1/z^{n+1}))} \\ &= e^{-4A/z} + O\big(1/z^{n+1}\big). \end{aligned} \qquad (25)$$

Now, using $P_n(x) = \prod_{i=1}^{n}(x - x_i)$ and the new transformation one gets

$$\begin{aligned} P_n\left(\tfrac{1}{2}(z + 1/z)\right) &= \prod_{i=1}^{n}\left(\tfrac{1}{2}(z + 1/z) - \tfrac{1}{2}(z_i + 1/z_i)\right) \\ &= \prod_{i=1}^{n}\frac{1}{2z}((z^2 + 1) - z(z_i + 1/z_i)) \\ &= \frac{1}{2^n z^n}\prod_{i=1}^{n}(z - z_i)(z - 1/z_i) \\ &= \frac{1}{2^n z^n}P_a(z)P_b(z). \end{aligned}$$

Therefore

$$\frac{P_a(z)P_b(z)}{P_a(-z)P_b(-z)} = (-1)^n\frac{P_n\left(\frac{1}{2}(z + 1/z)\right)}{P_n\left(-\frac{1}{2}(z + 1/z)\right)}.$$

Combining this with (25) and the identity (16) gives

$$G_n(1/x)\big|_{x=1/2(z+1/z)} = e^{-4A/z} + O\big(1/z^{n+1}\big)$$

from which we determine that $G_n(1/x) = G_A(1/x) + O(1/x^{2n+1})$ where $G_A(1/x)$ is given by

$$G_A(1/x) = e^{-4A(x - \sqrt{x^2-1})}.$$

The algorithm for computing g_i is of complexity $O(n)$ only. This algorithm follows the general power series algorithms described in [46], [47]. The key to this algorithm is to notice that $G_A(1/x)$ satisfies a second order linear differential equation (with singularities at $x = -1, 1, \infty$ and an apparent singularity at $x = 0$):

$$x(x^2 - 1)y'' + (8Ax(x^2 - 1) + 1)y' + 4A(1 - 4Ax)y = 0.$$

If we look at the expansion of $G_A(1/x)$ at $x = \infty$

$$G_A(1/x) = \sum_{n=0}^{\infty}\frac{g_n}{x^n}$$

then we get the fourth-order recurrence on g_n

$$g_{n+2} = \frac{(n^2 + 3n + 2 - 16A^2)g_{n+1} + A(8n + 4)g_n - (n^2 - 1)g_{n-1}}{8A(n + 2)}.$$

The initial conditions are $g_n = 0$ for $n < 0$ and

$$g_0 = 1, \quad g_1 = -2A, \quad g_2 = 2A^2, \ldots.$$

From these values and the equation for g_n one derives the C_n factors in the three-term recurrence for orthogonal polynomials $P_n(x)$. The first few C_n are

$$C_0 = -A, \quad C_1 = \frac{4A^2 - 3}{12}, \quad C_2 = \frac{45 - 60A^2 + 16A^4}{60(4A^2 - 3)}.$$

All C_n are rational functions in A of rather special structure. Since the case of the continued fraction expansion of $G_A(1/x)$ is not "explicitly solvable," C_n is not a "known" function of A and n; it belongs to the category of "higher Painleve" transendencies (see examples of such functions in [48]). An important consequence of the "insolubility" of C_n is the growth of coefficients of C_n as rational functions in A with integer coefficients. According to standard conjectures about explicit and nonexplicit continued fraction expansions (cf., [49]), the coefficients of C_n as a rational function in A over $\mathbf{Z}$ grow as $e^{O(n^2)}$ for large n. In fact, already for $n = 12$, the coefficients of C_n in A are large integers. This makes it impractical to precompute with full accuracy C_n for large n. It is also unnecessary to analytically determine $C_n = C_n(A)$ explicitly, since we need to know $C_n(A)$ only in the range of A that is significant for applications—this is the range where the weight of the orthogonal polynomials $P_n(x)$ is positive. In this case a much simpler best fit polynomial or rational approximation in A to $C_n(A)$ suffices.

The recursive algorithm for computing $P_n(x)$ can be summarized as follows.

Algorithm RECSOL: Given A and n, the polynomials $P_n(x)$ for be recursively computed as follows.

1) Set $g_0 = 1$, $g_1 = -2A$, $g_2 = 2A^2$ and for $k = 1$ to $2n-3$ let

$$g_{k+2} = \frac{(k^2+3k+2-16A^2)g_{k+1} + A(8k+4)g_k - (k^2-1)g_{k-1}}{8A(k+2)}.$$

2) Set $P_0(x) = 1$ and $P_1(x) = x - A$ and for $k = 1$ to $n-1$ let

$$C_k = -\frac{\sum_{i=0}^{k}(-1)^i g_{2k+1-i}p_{k,\,i}}{\sum_{i=0}^{k-1}(-1)^i g_{2k-1-i}p_{k-1,\,i}}$$
$$P_{k+1}(x) = xP_k(x) + C_kP_{k-1}(x).$$

Find the roots x_i of $P_n(x)$.

3) Set $\beta_i = \arccos x_i$, $i = 1, \ldots, n$, with $\beta_i \in (0, \pi)$. For $\beta_i \in (0, \pi/2)$, set $\alpha_i = \beta_i$. For $\beta_i \in (\pi/2, \pi)$, set $\alpha_i = \pi - \beta_i$. Sort the angles α_i.

Step 3 of algorithm RECSOL is the same as step 3 of algorithm HESOL. Using a computer algebra system, such as Maple or Mathematica, this recursive algorithm allows one to obtain $P_n(x)$ as an explicit function of A. A Maple program that implements algorithm RECSOL is given in the Appendix.

VI. Complexity of the PWM Problem

What is the complexity (in the simplest definition, an operation count) of the problem of "sums of odd powers"—of the optimal PWM problem discussed above? In the algebraic formulation one can first ask the same question about Newton's identities. If one uses Newton identities directly, the complexity is $O(n^2)$, but a significantly faster scheme can be found. The key to this is the representation (14). Indeed, according to Brent's theorem, N terms of the power series expansion of $e^{V(x)}$ can be computed in only $O(N \log N)$ steps from the power series expansion of $V(x)$ (see [43, Sec. 4.7, Example 4]). This algorithm requires only FFT-techniques for fast convolution [50].

Similar complexity considerations can be applied to the problem of fast computation of polynomials $P_n(x)$ that give the solution to the PWM problem of consecutive odd power sums. A naive method of computation of Padé approximants by solving dense systems of linear equations give a very high complexity bound of $O(n^3)$. A classical Levinson algorithm of solving systems of Toeplitz linear equations, or algorithms based on three-term recurrence relations satisfied by $P_m(x)$, give the complexity of computations of (all coefficients of) $P_n(x)$ at $O(n^2)$. These algorithms are, perhaps, the best for moderate n because of their simple nature and the fact that they use almost no additional space. Also, the $O(n^2)$ algorithm yields not only the single $P_n(x)$ but all $P_m(x)$ for $m \leq n$. However, for large n these algorithms became impractical and fast algorithms are needed.

A fast algorithm for computing (all coefficients of) $P_n(x)$ with total complexity of $O(n \log^2 n)$ works as follows. First, one applies Brent's theorem to compute $O(n)$ terms of the power series expansion (at infinity) of

$$G(1/x) = e^{-2\sum_{m\,\mathrm{odd}}^{\infty}(s_m/mx^m)}$$

from the first $O(n)$ terms s_m with the complexity of only $O(n \log n)$. Then, one uses fast Padé approximation algorithms (or equivalently, fast polynomial gcd algorithms). There is a variety of such algorithms (for the earliest of Knuth, see [51]), with the most popular belonging to Brent, Gustavson, and Yun [52] with $O(n \log^2 n)$ complexity. Thus, we can compute $P_n(x)$ in at most $O(n \log^2 n)$ operations. Of course, this method should be used only for a large n (with additional precision of calculations) since it relies on a variety of extensions of FFT methods that become advantageous only for very long arrays.

We wish to note also that in [22] Sun and Grotstollen present a Newton's iteration directly on the transformed variables x_i, and furthermore they note that the Jacobian in that case is a Vandermonde matrix, and that efficient $O(n^2)$ algorithms are available. [In fact $O(n \log^2 n)$ algorithms are available, which become efficient for large n]. However, Newton's iteration can be very sensitive to initial values.

VII. Conclusions

This paper presents a contribution to the theory of optimal pulse-width modulation and gives algorithms for efficient on-line calculation of PWM switching patterns. Some specific results for the harmonic elimination problem are also presented. A number of new results regarding the PWM problem are derived in this paper. It is shown that:

- by a transformation of variables the solution to the optimal PWM problem is given by the roots of a polynomial $P_n(x)$ that are appropriately sorted;

- the polynomials $P_n(x)$ satisfy

$$P_n(x) = (-1)^n P_n(-x) \exp\left\{-2 \sum_{\substack{1\le m\\ m \text{ odd}}} \frac{s_m}{m x^m}\right\}$$

where

$$s_m = \sum_{i=0}^{n} x_i^m$$

and x_i are the roots of n-degree monic polynomial $P_n(x)$;
- the polynomials $P_n(x)$ can be found by solving a Toeplitz system of linear equations;
- the polynomials $P_n(x)$ give also the solution to a Padé approximation problem and, therefore, constitute a set of orthogonal polynomials;
- the polynomials $P_n(x)$ are obtained through a simple recurrence

$$P_{n+1}(x) = x \cdot P_n + C_n \cdot P_{n-1}(x)$$

where an expression for C_n is given in (21);
- the complexity of a fast algorithm for the optimal PWM problem is $O(n \log^2 n)$.

An important consequence of the solution of the transcendental "sums of odd cosines" problem in PWM applications lies in the ability to construct high-quality digital (step-function) approximations to arbitrary harmonic series. In this paper, we have paid particular attention to the PWM harmonic elimination problem in which coefficients s_{2m-1} in (6) depend only on the fundamental component $h_1 = A$. In general, optimal PWM coefficients s_{2m-1} depend on the amplitudes of other harmonics h_{2k-1} which do not add to the complexity of the problem since system (6) is linear. Since the proposed algorithms can be executed quite fast on any processor with high-performance DSP capabilities, it opens a possibility of better on-the-fly construction of arbitrary (analog) waveforms using simple digital logic. Areas of application include not only power electronic converters but also, for instance, control of microelectromechanical systems and digital audio amplification. It should be noted that the design equations corresponding to optimal PWM with bi-level waveforms [1] have the same structure as (2) and (3), therefore the results presented in this paper are directly applicable.

While applying these basic algorithms, one should be aware of the well-known inherent ill-conditioning of the corresponding Toeplitz matrix and Padé approximation calculations. Techniques for the regularization of the basic algorithms presented here will be dealt with in detail in future publications. Future work will also focus on such practical applications as three-phase inverters which eliminates harmonics with orders divisible by three in (2) and (3).

A more detailed description of this work is given in the technical report [45].

APPENDIX A
FIRST FEW $P_n(x)$

Using the recursive algorithm HESOL, the polynomials $P_n(x)$ as explicit functions of A can be obtained for the harmonic elimination problem using computer algebra software. Here are the first few $P_n(x)$ for $n = 0, 1, 2, 3, 4$

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x - A\\
P_2(x) &= x^2 - Ax - 1/4 + 1/3A^2\\
P_3(x) &= x^3 - Ax^2 + \frac{1}{10}\,\frac{(16A^4 + 15 - 30A^2)x}{-3 + 4A^2}\\
&\quad - \frac{1}{60}\,\frac{A(45 - 60A^2 + 16A^4)}{-3 + 4A^2}\\
P_4(x) &= x^4 - Ax^3 + \frac{3}{28}\,\frac{(64A^6 + 560A^2 - 336A^4 - 315)x^2}{45 - 60A^2 + 16A^4}\\
&\quad - \frac{1}{42}\,\frac{A(64A^6 - 504A^4 + 1260A^2 - 945)x}{45 - 60A^2 + 16A^4} + \frac{1}{1680}\\
&\quad \cdot \frac{4725 - 12\,600A^2 + 10\,080A^4 - 2880A^6 + 256A^8}{45 - 60A^2 + 16A^4}.
\end{aligned}$$

APPENDIX B
MAPLE PROGRAM

The following Maple program implements algorithm RECSOL for solving the PWM harmonic elimination problem. Other programs are available from the authors.

```
# Specify n and A.
n := 10;
A := 0.6;
Digits := 20;  # or desired precision

# compute expansion of G_A(x)
g[0] := 1:
g[1] := -2 * A:
g[2] := 2 * A^2:
for k from 1 to 2 * n - 3 do
  gA := g[k - 1] * (k^2 - 1):
  gB := g[k] * A * (8 * k + 4);
  gC := g[k + 1] * (k^2 + 3 * k + 2 - 16 * A^2):
  g[k + 2] := (gC + gB - gA)/(8 * A * (k + 2)):
  g[k + 2] := simplify(g[k + 2]);
od:

# compute polynomials P_n(x)
P[0] := 1:
P[1] := x - A:
for k from 1 to n - 1 do
  num := sum((-1)^i * g[2 * k + 1 - i]
    *coeff(P[k], x, k - i), i = 0..k);
  den := sum((-1)^i * g[2 * k - 1 - i]
    *coeff(P[k - 1], x, k - 1 - i), i = 0..k - 1);
  C[k] := -num/den;
  C[k] := simplify(C[k]);
  P[k + 1] := x * P[k] + C[k] * P[k - 1];
  P[k + 1] := simplify(P[k + 1]);
```

```
od:

# compute roots x[i] and transform back to alpha[i]
x := fsolve(P[n]):
pi := evalf(Pi):
for i from 1 to n do
  beta[i] := arccos(x[i]):
  if beta[i] < pi/2 then
    alpha[i] := beta[i]:
  else
    alpha[i] := pi - beta[i]:
  fi;
od:

# sort alpha[i]
alpha := sort([seq(alpha[k], k = 1..n)]);

# check:
for i from 1 to n do
  sum(-(-1)^m * cos((2 * i - 1) * alpha[m]), m = 1..n);
od;
```

APPENDIX C
RELATION TO THE THEORY OF PADÉ APPROXIMATION

The solution to the modified Newton problem, where one has the first n sums of *odd* powers, can also be obtained using the theory of Padé approximation. Moreover, using the theory of Padé approximation, we find that the polynomials $P_n(x)$ are orthogonal polynomials with respect to a specific weighting function that depends on A only. Consequently, we can draw upon methods developed for orthogonal polynomials to develop numerically stable fast algorithms for computing both the polynomials $P_n(x)$ for $n \geq 0$ and their roots.

We start with the general relation between Padé approximations and the generalization of Newton relations between power and elementary symmetric functions. Let us look at the Padé approximation of the order (n, d) to the series $g(x)$ at $x = \infty$. Here the function $g(x)$ is defined via the generating function of the sequence s_m: $g(x) = e^{-\sum_{m=1}^{\infty}(s_m/mx^m)}$. The Padé approximation of the order (n, d) to $g(x)$ at (the neighborhood of) $x = \infty$ is a rational function $P_n(x)/Q_d(x)$ with $P_n(x)$ a polynomial of degree n and $Q_d(x)$ a polynomial of degree d—such that the expansion of $P_n(x)/Q_d(x)$ matches the expansion of $g(x)$ at $x = \infty$ up to the maximal order. This means that

$$\frac{P_n(x)}{Q_d(x)} - x^{n-d}g(x) = O\left(x^{-2d-1}\right)$$

or

$$P_n(x) - Q_d(x)x^{n-d}g(x) = O\left(x^{-d-1}\right).$$

After taking the logarithmic derivative of this definition, we obtain the following representation of this definition:

$$\frac{P_n'}{P_n} - \frac{Q_d'}{Q_d} = \frac{d}{dx}\log x^{n-d}g(x) + O\big(x^{-n-d-2}\big).$$

Now, if we write the normalized (monic) polynomials $P_n(x)$ and $Q_d(x)$ in terms of their roots

$$P_n(x) = \prod_{i=1}^{n}(x - x_i); \quad Q_d(x) = \prod_{k=1}^{d}(x - y_k)$$

we get an identification of symmetric functions in x_i and y_k with the sequence of s_m in the definition of $g(x) = e^{-\sum_{m=1}^{\infty}(s_m/mx^m)}$. Namely, we get

$$\sum_{i=1}^{n} x_i^j - \sum_{k=1}^{d} y_k^j = s_j$$

for $j = 0, \ldots, n + d$ (where $s_0 = n - d$).

In the case $d = 0$, one simply recovers Newton's identities. The case $d = n$ (the "diagonal" Padé approximations) is the case that solves the problem of "sums of odd powers": Consider the "anti-symmetric" case when $y_i = -x_i$ for $i = 1 \cdots n$ and $d = n$. In this case, in the notations above, $s_{2m} = 0$ for $m \geq 0$. We, therefore, obtain the Padé approximation of order (n, n) to the following function:

$$G(1/x) = e^{-2\sum_{m \text{ odd}}^{\infty}(s_m/mx^m)}.$$

The Padé approximants $P_n(x)/Q_n(x)$ are related in this case by $Q_n(x) = (-1)^n P_n(-x)$. This relation between the numerator and the denominator of the Padé approximants to $G(1/x)$ must hold because $G(1/x)$ satisfies the simple functional identity: $G(-x) = 1/G(x)$. This gives the main result:

Theorem 1: The solution $\{x_i\}$ ($i = 1 \cdots n$) to the problem of "sums of odd powers"

$$\sum_{i=1}^{n} x_i^{2m-1} = s_{2m-1}, \qquad m = 1 \cdots n$$

is given by the roots of the numerator $P_n(x) = \prod_{i=1}^{n}(x - x_i)$ of the Padé approximation of order (n, n) to the function

$$G(1/x) = e^{-2\sum_{m \text{ odd}}^{\infty}(s_m/mx^m)}.$$

Another way to verify this approximation without specialization from the case of general sequence s_m, is to take the identity (16).

Proof of Theorem 1: First of all, the Padé approximation rational function $P_n(x)/Q_n(x)$ of order (n, n) is unique. [Indeed, if there would be two rational approximations p_1/q_1 and p_2/q_2 that approximate $g(x)$ up to x^{-2n-1}, then $(p_1/q_1) - (p_2/q_2) = O(x^{-2n-1})$, and since degrees of p_1, q_1, p_2, q_2 are bounded by n, then $p_1/q_1 = p_2/q_2$.] Then, if $P_n(x)/Q_n(x)$ is a Padé approximation of order (n, n) to $G(1/x)$, we assume that this representation of the rational function is irreducible [i.e., that $P_n(x)$ and $Q_n(x)$ are relatively prime]. Then, $Q_n(x)/P_n(x)$ is a Padé approximation of order (n, n) to $1/G(1/x)$, and $P_n(-x)/Q_n(-x)$ is a Padé approximation of order (n, n) to $G(-1/x)$. Because of the functional equation $G(-x) = 1/G(x)$, and the uniqueness of the Padé approximations, we get $Q_n(x)/P_n(x) = P_n(-x)/Q_n(-x)$. This equation means that $Q_n(x) = \alpha P_n(-x)$. Moreover, since the expansion of $G(1/x)$ at $x = \infty$ starts at 1 [i.e., $G(1/x) \to 1$ as $x \to \infty$], we have $P_n(x)/Q_n(x) \to 1$ as $x \to \infty$. This means that $\alpha = (-1)^n$ and $Q_n(x) = (-1)^n P_n(-x)$. Taking into account the above-mentioned "main" identity

$$(-1)^n \cdot \frac{P_n(x)}{P_n(-x)} = e^{-2\sum_{m\,\mathrm{odd}}^{\infty}(x^{-m}/m)\sum_{i=1}^{n} x_i^m}$$

we see that the right-hand side of this identity and the expansion of $G(1/x)$ at $x = \infty$ must agree up to (but not including) x^{-2n-1}. This means we have $\sum_{i=1}^{n} x_i^{2m-1} = s_{2m-1}$ for $m = 1 \cdots n$.

Notice that in the definition of $G(1/x)$, the value(s) of s_{2m-1} for $m > n$ have no impact on the definition of $P_n(x)$ because they enter the expansion of $G(1/x)$ only at x^{-k} for $k \geq 2n + 1$. This is obvious because $e^{\sum_{k=1}^{\infty} \beta_k x^{-k}} = e^{\sum_{k=1}^{m} \beta_k x^{-k}} + O(x^{-m-1})$.

This completes the proof of Theorem 1.

Since we identified the solution to the "sums of odd powers" problem with numerator (or denominator) in the (diagonal) Padé approximation problem, we infer from the standard theory of continued fraction expansions that the rational functions $P_n(x)/P_n(-x)$ are partial fractions in the continued fraction expansion of the generating function $G(1/x)$ at $x = \infty$. This also means (see Szego for these and other facts of the theory of continued fraction expansions and orthogonal polynomials [53]) that the sequence of polynomials $P_n(x)$ is the sequence of orthogonal polynomials [with respect to the weight that is Hilbert transform of $G(1/z)$], and that the sequence of polynomials $P_n(x)$ satisfies three-term linear recurrence relation. Since the same recurrence is satisfied both by numerators and denominators of the partial fractions, the recurrence is satisfied by two sequences—$P_n(x)$ and $(-1)^n \cdot P_n(-x)$. With the leading coefficient of $P_n(x)$ is 1, one obtains the particularly simple three-term recurrence relation among $P_n(x)$

$$P_{n+1}(x) = x \cdot P_n(x) + C_n \cdot P_{n-1}(x)$$

for $n \geq 0$, which was introduced above where it was developed using the matrix notation.

ACKNOWLEDGMENT

The authors wish to thank H. Huang for helpful comments on the manuscript.

REFERENCES

[1] P. Enjeti, P. D. Ziogas, and J. F. Lindsay, "Programmed PWM techniques to eliminate harmonics: A critical evaluation," *IEEE Trans. Ind. Applicat.*, vol. 26, pp. 302–316, Mar. 1990.

[2] A. M. Trzynadlowski, "An overview of modern PWM techniques for three-phase, voltage-controlled, voltage-source inverters," in *Proc. IEEE Symp. Industrial Electronics ISIE'96*, 1996, pp. 25–39.

[3] J. Holtz, "Pulsewidth modulation—A survey," *IEEE Trans. Ind. Electron.*, vol. 39, pp. 410–420, Oct. 1992.

[4] H. S. Patel and R. G. Hoft, "Generalized technique of harmonic elimination and voltage control in thyristor inverters: Part I harmonic elimination," *IEEE Trans. Ind. Applicat.*, pp. 310–317, May 1973.

[5] P. Enjeti and J. F. Lindsay, "Solving nonlinear equation of harmonic elimination PWM in power control," *Electron. Lett.*, vol. 23, no. 12, pp. 656–657, 1987.

[6] S. R. Bowes and R. I. Bullough, "Optimal PWM microprocessor-controlled current-source inverter drives," *Proc. Inst. Elect. Eng. B*, vol. 135, no. 2, pp. 59–75, 1988.

[7] J. Richardson and S. V. S. Mohan-Ram, "An auto-adjusting PWM waveform generation technique," in *Proc. Conf. Power Electronics and Variable-Speed Drives*, 1988, pp. 366–369.

[8] S. R. Bowes and R. I. Bullough, "Novel PWM controlled series-connected current-source inverter drive," *Proc. Inst. Elect. Eng. B*, vol. 136, no. 2, pp. 69–82, 1989.

[9] K. E. Addoweesh and A. L. Mohamadein, "Microprocessor based harmonic elimination in chopper type AC voltage regulators," *IEEE Trans. Power Electron.*, vol. 5, pp. 191–200, Apr. 1990.

[10] Q. Jiang, D. G. Holmes, and D. B. Giesner, "A method for linearising optimal PWM switching strategies to enable their computation on-line in real-time," in *Proc. IEEE Industry Applications Soc. Annu. Meeting*, vol. 1, 1991, pp. 819–825.

[11] T. Xinyuan and B. Jingming, "An algebraic algorithm for generating optimal PWM waveforms for AC drives—Part I: Selected harmonic elimination," in *Rec. IEEE Power Electronics Specialists Conf. (PESC'91)*, 1991, pp. 402–408.

[12] S. R. Bowes and P. R. Clark, "Simple microprocessor implementation of new regular-sampled harmonic elimination PWM techniques," *IEEE Trans. Ind. Applicat.*, vol. 28, pp. 89–95, Jan. 1992.

[13] P. N. Enjeti and W. Shireen, "A new technique to reject DC-link voltage ripple for inverters operating on programmed PWM waveforms," *IEEE Trans. Power Electron.*, vol. 7, pp. 171–180, Jan. 1992.

[14] A. J. Frazier and M. K. Kazimierczuk, "Dc–dc power inversion using sigma–delta modulation," *IEEE Trans. Circuits Syst. I*, vol. 46, pp. 79–82, Jan. 2000.

[15] A. Wang and S. R. Sanders, "On optimal programmed PWM waveforms for dc–dc converters," in *Rec. IEEE Power Electronics Specialists Conference (PESC'92)*, 1992, pp. 571–578.

[16] S. R. Bowes and P. R. Clark, "Transputer-based harmonic-elimination PWM control of inverter drives," *IEEE Trans. Ind. Applicat.*, vol. 28, pp. 72–80, Jan. 1992.

[17] E. Destobbeleer and D. Dubois, "Solving a nonlinear and incomplete equation system for a programmed PWM: Application to electric traction," in *Proc. IEE Colloq. Variable Speed Drives and Motion Control*, 1992, pp. 5/1–5/5.

[18] J. Sun and H. Grotstollen, "Solving nonlinear equations for selective harmonic eliminated PWM using predicted initial values," in *Proc. Int. Conf. Industrial Electr., Control and Instr.*, 1992, pp. 259–264.

[19] F. Swift and A. Kamberis, "Walsh domain technique of harmonic elimination and voltage control in pulse-width modulated inverters," *IEEE Trans. Power Electron.*, vol. 8, pp. 170–185, Apr. 1993.

[20] U. S. Sapre, N. K. Ashar, and S. A. Joshi, "A new approach to programmed PWM implementation," in *Proc. Conf. Appl. Power Electronics Conf. and Expos. (APEC'93)*, 1993, pp. 793–798.

[21] J. Sun, S. Beineke, and H. Grotstollen, "DSP-based real-time harmonic elimination of PWM inverters," in *Rec. IEEE Power Electronics Specialists Conf. (PESC'94)*, 1994, pp. 679–685.

[22] J. Sun and H. Grotstollen, "Pulsewidth modulation based on real-time solution of algebraic harmonic elimination equations," in *Proc. Int. Conf. Industrial Electronics, Control and Instrumentation, (IECON'94)*, 1994, pp. 79–84.

[23] H. R. Karshenas, H. A. Kojori, and S. B. Dewan, "Generalized techniques of selective harmonic elimination and current control in current source invertersconverters," *IEEE Trans. Power Electron.*, vol. 10, pp. 566–573, Sept. 1995.

[24] J. X. Lee and W. J. Bonwick, "Solving nonlinear cost function for optimal PWM strategy," in *Proc. Int. Conf. Power Electronics and Drive Systems*, vol. 1, 1995, pp. 395–400.

[25] H. L. Liu, G. H. Cho, and S. S. Park, "Optimal PWM design for high power three-level inverter through comparative studies," *IEEE Trans. Power Electron.*, vol. 10, pp. 38–47, Jan. 1995.

[26] J. Sun, S. Beineke, and H. Grotstollen, "Optimal PWM based on real-time solution of harmonic elimination equations," *IEEE Trans. Power Electron.*, vol. 11, pp. 612–621, July 1996.

[27] J. Richardson, G. J. Bezanov, and G. M. Hashem, "Optimal PWM waveform modeling for on-line ac drive control," in *Proc. Conf. Power Electronics and Variable Speed Drives*, 1996, pp. 496–501.

[28] G. Bezanov and J. Richardson, "Algorithmic modeling of PWM solution sets for online control of inverters," in *Proc. Power Electronics Specialists Conf. (PESC '92)*, 1996, pp. 595–600.

[29] T.-J. Liang, R. M. O'Connell, and R. G. Hoft, "Inverter harmonic reduction using Walsh function harmonic elimination method," *IEEE Trans. Power Electron.*, vol. 12, pp. 971–982, Nov. 1997.

[30] J. W. Chen and T. J. Liang, "A novel algorithm in solving nonlinear equations for programmed PWM inverter to eliminate harmonics," in *Proc. Int. Conf. Industrial Electronics, Control and Instrumentation (IECON'97)*, vol. 2, 1997, pp. 698–703.

[31] H. Han, Z. Hui, H. Yongxuan, and J. Yanchao, "A new optimal programmed PWM technique for power convertors," in *Proc. Int. Conf. Power Electronics and Drive Systems*, vol. 2, 1997, pp. 740–743.

[32] R. Pindado, C. Jaen, and J. Pou, "Robust method for optimal PWM harmonic elimination based on the Chebyshev functions," in *Proc. 8th Int. Conf. on Harmonics and Quality of Power*, vol. 2, 1998, pp. 976–981.
[33] J. Singh, "Performance analysis of microcomputer based PWM strategies for induction motor drive," in *Proc. IEEE Int. Symp. Industrial Electronics (ISIE'98)*, vol. 1, 1998, pp. 277–282.
[34] A. I. A. Maswood and L. Jian, "Optimal online algorithm derivation for PWM-SHE switching," *Electron. Lett.*, vol. 34, no. 8, pp. 821–823, 1998.
[35] G.-H. Choe and M.-H. Park, "A new injection method for AC harmonic elimination by active power filter," *IEEE Trans. Ind. Electron.*, vol. 35, pp. 141–147, Feb. 1988.
[36] T. Kato, "Sequential homotopy-based computation of multiple solutions for selected harmonic elimination in PWM inverters," *IEEE Trans. Circuits Syst. I*, vol. 46, pp. 586–593, May 1999.
[37] J. W. Dixon, J. M. Contardo, and L. A. Moran, "A fuzzy-controlled active front-end rectifier with current harmonic filtering characteristics and minimum sensing variables," *IEEE Trans. Power Electron.*, vol. 14, pp. 724–729, July 1999.
[38] L. Li, D. Czarkowski, Y. Liu, and P. Pillay, "Suppression of harmonics in multilevel series-connected PWM inverters," in *Proc. IEEE IAS Annual Meeting*, 1998, pp. 1454–1461.
[39] S. R. Bowes and P. R. Clark, "Regular-sampled harmonic-elimination PWM control of inverter drives," *IEEE Trans. Power Electron.*, vol. 10, pp. 521–531, Sept. 1995.
[40] S. R. Bowes and S. Grewal, "A novel harmonic elimination PWM strategy," in *Proc. Conf. Power Electronics and Variable Speed Drives*, Sept. 1998, pp. 426–432.
[41] ——, "Simplified harmonic elimination PWM control strategy," *Electron. Lett.*, vol. 34, no. 4, pp. 325–326, 19, 1998.
[42] W. Ledermann, *Introduction to Group Character*. Cambridge, MA: Cambridge Univ. Press, 1977.
[43] D. E. Knuth, *The Art of Computer Programming*. New York: Addison-Wesley, 1981, vol. 2.
[44] S. Zohar, "The solution of a Toeplitz set of linear equations," *J. Assoc.Comput. Mach.*, vol. 21, no. 2, pp. 272–276, Apr. 1974.
[45] D. Czarkowski, D. Chudnovsky, G. Chudnovsky, and I. Selesnick, "Solving the optimal PWM problem for single-phase inverters," Polytechnic Univ., Brooklyn, NY, Tech. Rep., 1999.
[46] D. V. Chudnovsky and G. V. Chudnovsky, "On expansion of algebraic functions in power and Puiseux series, I," *J. Complexity*, vol. 2, no. 4, pp. 271–294, 1986.
[47] ——, "On expansion of algebraic functions in power and Puiseux series, II," *J. Complexity*, vol. 3, no. 1, pp. 1–25, 1987.
[48] ——, "Explict continued fractions and quantum gravity," *Acta Applic. Math*, vol. 36, pp. 167–185, 1994.
[49] ——, "Transcendental methods and theta-functions," in *Proc. Symp. Pure Math.*, vol. 2, 1989, pp. 167–232.
[50] A. B. Borodin and I. Munro, *The Computational Complexity of Algebraic and Numeric Problems*, NY: American Elsevier, 1975.
[51] D. Knuth, "The analysis of algorithms," *Actes Congrès. Intern. Math.*, vol. 3, pp. 269–274, 1970.
[52] R. P. Brent, F. G. Gustavson, and D. Y. Yun, "Fast solution of Toeplitz systems of equations and computation of Padé approximants," *J. Algorithms*, vol. 1, no. 3, pp. 259–295, Sept. 1980.
[53] G. Szego, *Orthogonal Polynomials*. Providence, RI: AMS, 1978.

Dariusz Czarkowski (M'97) received the M.S. degrees in electronics and electrical engineering from the University of Mining and Metallurgy, Cracow, Poland, and from Wright State University, Dayton, OH, in 1989 and 1993, respectively, and the Ph.D. degree in electrical engineering from the University of Florida, Gainesville, FL, in 1996.

He joined the Polytechnic University, Brooklyn, NY, as an Assistant Professor of electrical engineering in 1996. His research interests are in the areas of power electronics, electric drives, and power quality. He coauthored a book entitled *Resonant Power Converters* (New York: Wiley Interscience, 1995).

David V. Chudnovsky (M'00) received the Ph.D. degree from the Institute of Mathematics of Ukranian Academy of Science, Kiev, Ukraine, in 1972.

He was Senior Research Scientist in the Department of Mathematics at Columbia University, New York, from 1978 to 1995, and a Research Scientist with CNRS and Ecole Polytechnique Paris, France, from 1977 to 1981. Since 1996, he is a Distinguished Industry Professor of Mathematics, Department of Mathematics, Polytechnic University, Brooklyn, New York, and the Director of Institute for Mathematics and Advanced Supercomputing (IMAS), Brooklyn, New York. His research interests include: partial differential equations and Hamiltonian systems, quantum systems, computer algebra and complexity, large scale numerical mathematics, parallel computing and digital signal processing. Dr. Chudnovsky is a member of ACM and APS.

Gregory V. Chudnovsky received the Ph.D. degree from the Institute of Mathematics of Ukranian Academy of Science, Kiev, Ukraine, in 1975.

He was Senior Research Scientist in the Department of Mathematics at Columbia University, New York, from 1978 to 1995 and a Research Scientist with CNRS at IHES, Paris, France, from 1977 to 1981. Since 1996, he is a Distinguished Industry Professor of Mathematics, Department of Mathematics, Polytechnic University and Director of Institute for Mathematics and Advanced Supercomputing (IMAS), Brooklyn, NY. His research interests include: number theory, mathematical physics, computer algebra and complexity, large scale numerical mathematics, parallel computing and digital signal processing. Dr. Chudnovsky is a member of APS.

Ivan W. Selesnick (S'91–M'95) received the B.S., M.E.E., and Ph.D. degrees in electrical engineering, from Rice University, Houston, TX, in 1990, 1991, and 1996, respectively.

In 1997, he was a visiting professor at the University of Erlangen, Nurnberg, Germany. He is currently an Aassistant Professor in the Department of Electrical and Computer Engineering at Polytechnic University, Brooklyn, NY. His current research interests are in the area of digital signal processing and wavelet-based signal processing.

As a Ph.D. student Dr. Selesnick received a DARPA-NDSEG fellowship in 1991, and his Ph.D. dissertation received the Budd Award for Best Engineering Thesis at Rice University in 1996 and an award from the Rice-TMC chapter of Sigma Xi. He received an Alexander von Humboldt Award in 1997 and a National Science Foundation Career award in 1999. He is a member of Eta Kappa Nu, Phi Beta Kappa, Tau Beta Phi.