The Design of Approximate Hilbert Transform Pairs of Wavelet Bases

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***Abstract*—Several authors have demonstrated that significant improvements can be obtained in wavelet-based signal processing by utilizing a pair of wavelet transforms where the wavelets form a Hilbert transform pair. This paper describes design procedures, based on spectral factorization, for the design of pairs of dyadic wavelet bases where the two wavelets form an approximate Hilbert transform pair. Both orthogonal and biorthogonal FIR solutions are presented, as well as IIR solutions. In each case, the solution depends on an allpass filter having a flat delay response. The design procedure allows for an arbitrary number of vanishing wavelet moments to be specified. A Matlab program for the procedure is given, and examples are also given to illustrate the results.**

***Index Terms*—Dual-tree complex wavelet transform, Hilbert transform, wavelet transforms.**

I. Introduction

THIS paper describes design procedures, based on spectral factorization, for the design of pairs of dyadic wavelet bases where the two wavelets form an approximate Hilbert transform pair. Several authors have advocated the simultaneous use of two wavelet transforms where the wavelets are so related. For example, Abry and Flandrin suggested using a Hilbert pair of wavelets for transient detection [2] and turbulence analysis [1]. Ozturk *et al.* suggested it for waveform encoding [14]. They are also useful for implementing complex and directional wavelet transforms. Freeman and Adelson employ the Hilbert transform in the development of steerable filters [5], [20]. Kingsbury's complex dual-tree DWT [8], [9] is based on (approximate) Hilbert pairs of wavelets. The steerable pyramid and the dual-tree DWT have numerous benefits, including improved denoising capability and the fact that they are both directional and nearly shift-invariant. The paper by Beylkin and Torrésani [3] is also of related interest.

One could start with a known wavelet and then take its Hilbert transform to obtain the second wavelet; however, in that case, the second wavelet would not be of finite support. One could design a finitely supported wavelet to approximate the infinitely supported Hilbert transform, but in this paper, we design both wavelets together to better utilize the degrees of freedom.

Using the infinite product formula, it was shown in [18] that for two orthogonal wavelets to form a Hilbert transform pair, the scaling filters should be offset by a half sample. In [18], a design problem was formulated for the minimal length scaling filters such that 1) the wavelets each have a specified number of vanishing moments (K), and 2) the half-sample delay approximation is flat at $\omega = 0$ with specified degree (L). However, this formulation leads to nonlinear design equations, and the examples in [18] had to be obtained using Gröbner bases. In this paper, we describe a design procedure based on spectral factorization. It results in filters similar to those of [18]; however, the design algorithm is much simpler and more flexible.

A. Preliminaries

Let the filters $h_0(n), h_1(n)$ represent a CQF pair [22]. That is

$$\sum_n h_0(n)h_0(n+2k) = \delta(k) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$$

and $h_1(n) = (-1)^n h_0(M-n)$, where M is an odd integer. Equivalently, in terms of the Z-transform, we have

$$H_0(z)H_0(1/z) + H_0(-z)H_0(-1/z) = 2$$

and

$$H_1(z) = (-z)^{-M} H_0(-1/z).$$

Let the filters $g_0(n), g_1(n)$ represent a second CQF pair. In this paper, we assume $h_i(n), g_i(n)$ are real-valued filters. It is convenient to write the CQF condition in terms of the autocorrelation functions defined as

$$p_h(n) := \sum_k h_0(k)h_0(k-n) = h_0(n) * h_0(-n)$$
$$p_g(n) := \sum_k g_0(k)g_0(k-n) = g_0(n) * g_0(-n)$$

or equivalently as

$$P_h(z) := H_0(z)H_0(1/z)$$
$$P_g(z) := G_0(z)G_0(1/z).$$

Then, $h_0(n)$ and $g_0(n)$ satisfy the CQF conditions if and only if $p_h(n)$ and $p_g(n)$ are halfband filters:

$$p_h(n) = \begin{cases} 1, & n=0 \\ 0, & n = \pm 2, \pm 4, \ldots \end{cases}$$

and similarly for $p_g(n)$. This can be written more compactly as

$$p_h(2n) = \delta(n) \quad \text{and} \quad p_g(2n) = \delta(n). \tag{1}$$

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The dilation and wavelet equations give the scaling and wavelet functions

$$\phi_h(t) = \sqrt{2}\sum_n h_0(n)\phi_h(2t-n) \tag{2}$$

$$\psi_h(t) = \sqrt{2}\sum_n h_1(n)\phi_h(2t-n). \tag{3}$$

The scaling function $\phi_g(t)$ and wavelet $\psi_g(t)$ are defined similarly but with filters $g_0(n)$ and $g_1(n)$.

Notation: The Z-transform of $h(n)$ is denoted by $H(z)$. The discrete-time Fourier transform of $h(n)$ will be denoted by $H(\omega)$, although it is an abuse of notation. The Fourier transform of $\psi(t)$ is denoted by $\Psi(\omega) = \mathcal{F}\{\psi(t)\}$.

B. Hilbert Transform Pairs

In [18], it was shown that if $H_0(\omega)$ and $G_0(\omega)$ are lowpass CQF filters with

$$G_0(\omega) = H_0(\omega)e^{-j(\frac{\omega}{2})} \quad \text{for } |\omega| < \pi \tag{4}$$

then the corresponding wavelets form a Hilbert transform pair

$$\psi_g(t) = \mathcal{H}\{\psi_h(t)\}.$$

That is

$$\Psi_g(\omega) = \begin{cases} -j\Psi_h(\omega), & \omega > 0 \\ j\Psi_h(\omega), & \omega < 0. \end{cases}$$

C. Flat-Delay Allpass Filter

The design procedure presented in this paper depends on the design of an allpass filter with approximately constant fractional delay. Several authors have addressed the design of allpass systems that approximate a fractional delay [11], [12], [16]. The following formula for the maximally flat delay allpass filter is adapted from Thiran's formula for the maximally flat delay allpole filter [23]. The maximally flat approximation to a delay of τ samples is given by

$$A(z) = \frac{z^{-L}D(1/z)}{D(z)}$$

where

$$D(z) = 1 + \sum_{n=1}^{L} d(n)z^{-n}$$

with

$$d(n) = (-1)^n \binom{L}{n} \frac{(\tau - L)_n}{(\tau+1)_n} \tag{5}$$

where $(x)_n$ represents the rising factorial

$$(x)_n := \underbrace{(x)(x+1)\ldots(x+n-1)}_{n \text{ terms}}.$$

With this $D(z)$, we have the approximation

$$A(z) \approx z^{-\tau} \quad \text{around } z = 1$$

or equivalently

$$A(\omega) \approx e^{-j\omega\tau} \quad \text{around } \omega = 0.$$

The coefficients $d(n)$ in (5) can be computed very efficiently using the following ratio.

$$\frac{d(n+1)}{d(n)} = -\frac{\binom{L}{n+1}}{\binom{L}{n}} \cdot \frac{(\tau-L)_{n+1}}{(\tau-L)_n} \cdot \frac{(\tau+1)_n}{(\tau+1)_{n+1}} = \frac{(L-n)(L-n-\tau)}{(n+1)(n+1+\tau)}.$$

From this ratio, it follows that the filter $d(n)$ can be generated as follows:

$$d(0) = 1$$
$$d(n+1) = d(n) \cdot \frac{(L-n)(L-n-\tau)}{(n+1)(n+1+\tau)}, \quad 0 \le n \le L-1.$$

This can be implemented in Matlab with the commands

```
n = 0:L-1;
r = [1, (L-n).*(L-n-tau)./(n+1)./(n+1+tau)];
d = cumprod(r) .
```

In our problem, we will use $d(n)$ in (5) with $\tau = 1/2$. For example, with $L = 2$, we have

$$d(n) = \{1, 2, 1/5\}, \quad \text{for } n = 0, 1, 2.$$

With $L = 3$, we have

$$d(n) = \{1, 5, 3, 1/7\}, \quad \text{for } n = 0, 1, 2, 3.$$

II. Orthogonal Solutions

In this section, we look for pairs of orthonormal wavelets where the lowpass scaling filters have the form

$$h_0(n) = f(n) * d(n)$$
$$g_0(n) = f(n) * d(L-n)$$

where the filter $d(n)$ will be chosen to achieve the (approximate) half-sample delay. The first step of the design procedure will be to determine the appropriate filter $d(n)$ to achieve the desired relationship between $h_0(n)$ and $g_0(n)$. In terms of the transfer functions, we have

$$H_0(z) = F(z)D(z)$$
$$G_0(z) = F(z)z^{-L}D(1/z).$$

$H_0(z)$ and $G_0(z)$ have the common divisor $F(z)$, which will be determined later. We can write

$$G_0(z) = H_0(z)\frac{z^{-L}D(1/z)}{D(z)}$$

where we can recognize that the transfer function

$$A(z) := \frac{z^{-L}D(1/z)}{D(z)}$$

is an allpass system $|A(\omega)| = 1$. Therefore

$$|G_0(\omega)| = |H_0(\omega)|, \quad |G_1(\omega)| = |H_1(\omega)|$$

and

$$|\Psi_g(\omega)| = |\Psi_h(\omega)|.$$

If the allpass system $A(z)$ is an approximate half-sample delay

$$A(\omega) \approx e^{-j\omega/2} \quad \text{around } \omega = 0$$

or equivalently, $A(z) \approx z^{-1/2}$ around $z = 1$, then the sought-after approximation (4) is achieved

$$G_0(\omega) \approx H_0(\omega) e^{-j\frac{\omega}{2}} \quad \text{around } \omega = 0.$$

This is achieved by taking the allpass filter $d(n)$ in (5) with $\tau = 1/2$. (The point $z = 1$ is chosen for the point of approximation for the lowpass filter because that is the center of the passband.)

To obtain wavelet bases with K vanishing moments, we let

$$F(z) = Q(z)(1 + z^{-1})^K.$$

Then

$$H_0(z) = Q(z)(1 + z^{-1})^K D(z) \tag{6}$$
$$G_0(z) = Q(z)(1 + z^{-1})^K z^{-L} D(1/z). \tag{7}$$

We now have the following design problem. Given $D(z)$ and K, find $Q(z)$ of minimal degree such that $h_0(n)$ and $g_0(n)$ satisfy the CQF conditions (1). Note that with (6) and (7), $h_0(n)$ and $g_0(n)$ have the same autocorrelation function

$$\begin{aligned} P(z) &:= P_h(z) = P_g(z) \\ &= Q(z)Q(1/z)(z + 2 + z^{-1})^K D(z) D(1/z). \end{aligned}$$

Similar to the way Daubechies wavelet filters are obtained, we can obtain $Q(z)$ using a spectral factorization approach as in [22]. The procedure consists of two steps.

1) Find $r(n)$ of minimal length such that
   a) $r(n) = r(-n)$, and
   b) $R(z)(z + 2 + z^{-1})^K D(z) D(1/z)$ is halfband.

   Note that $r(n)$ of minimal length will be supported on the range $(1 - K - L) \le n \le (K + L - 1)$.
2) Set $Q(z)$ to be a spectral factor of $R(z)$

$$R(z) = Q(z)Q(1/z). \tag{8}$$

To carry out the first step, we need only solve a system of linear equations. Defining

$$S(z) := (z + 2 + z^{-1})^K D(z) D(1/z)$$

we can write the halfband condition as

$$\begin{aligned} \delta(n) &= [\downarrow 2](s * r)(n) \\ &= \sum_k s(2n - k) r(k). \end{aligned}$$

When written in matrix form, this calls for a square matrix of dimension $2(K + L) - 1$, which has the form of a convolution (Toeplitz) matrix with every second row deleted.

TABLE I
MATLAB PROGRAM

```
function [h,g] = hwlet(K,L)
% Approximate Hilbert pair of orthogonal wavelets
% h, g - scaling filters of length 2*(K+L)
% K - number of zeros at z=-1
% L - degree of fractional delay

n = 0:L-1;
tau = 1/2;
d = cumprod([1, (L-n).*(L-n-tau)./(n+1)./(n+1+tau)]);
s1 = binom(2*K,0:2*K);
s2 = conv(d,d(end:-1:1));
s = conv(s1,s2);
M = K+L;
C = convmtx(s',2*M-1);
C = C(2:2:end,:);
b = zeros(2*M-1,1);
b(M) = 1;
r = (C\b)';
q = sfact(r);
f = conv(q,binom(K,0:K));
h = conv(f,d);
g = conv(f,d(end:-1:1));
```

The second step assumes $R(z)$ permits spectral factorization, which we have found to be true in all our examples. With $Q(z)$ obtained in this way, the filters $H_0(z)$ and $G_0(z)$ defined in (6) and (7) satisfy the CQF conditions and have the desired half-sample delay. Note that $Q(z)$ is not unique.

This design procedure yields filters $h_0(n)$ and $g_0(n)$ of (minimal) length $2(L + K)$. A Matlab program to implement this design procedure is given Table I. The commands `binom` and `sfact` for computing binomial coefficients and performing spectral factorization are not currently standard Matlab commands. They are available from the author.

Example 1A: With $K = 4$ and $L = 2$, the filters $h_0(n)$ and $g_0(n)$ are of length 12. Fig. 1 illustrates the solution obtained from a mid-phase spectral factorization. The plot of the function $|\Psi_h(\omega) + j\Psi_g(\omega)|$ shows that it approximates zero for $\omega < 0$, as expected if ψ_h and ψ_g form a Hilbert transform pair. The coefficients are given in Table II. The Sobolev exponent (reflecting the smoothness of the wavelets) was found[1] to be 1.983. Note that the Sobolev exponent is the same for $\psi_h(t)$ and $\psi_g(t)$.

Example 1B: We set $K = 4$ and $L = 2$ again, but this time, we take a minimum-phase spectral factor rather than a mid-phase one. The wavelets obtained in this case are illustrated in Fig. 2. The function $|\Psi_h(\omega) + j\Psi_g(\omega)|$ is exactly the same as in Example 1A; using a different spectral factor in (8) does not change it. We will see below that the mid-phase and minimum-phase solutions can lead to different results when they are used to implement two-dimensional (2-D) directional wavelet transforms. (The Sobolev exponent is the same as for Example 1A.)

Example 2: With $K = 3$ and $L = 3$, the filters $h_0(n)$ and $g_0(n)$ are again of length 12. Fig. 3 illustrates a solution using a mid-phase spectral factor. It can be seen that $|\Psi_h(\omega) + j\Psi_g(\omega)|$

[1]The Sobolev exponents were computed using the Matlab program `sobexp` by Ojanen (http://www.math.rutgers.edu/~ojanen/).

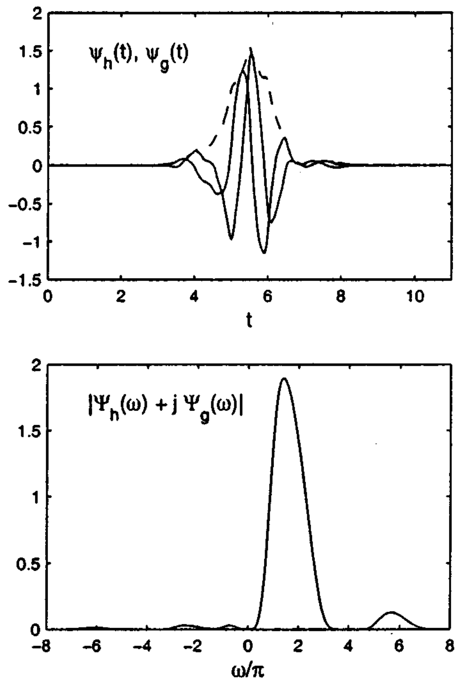

Fig. 1. Example 1A. Approximate Hilbert transform pair of orthonormal wavelet bases with $N = 12, K = 4, L = 2$, and mid-phase spectral factorization.

TABLE II
COEFFICIENTS FOR EXAMPLE 1A AND EXAMPLE 2

Example 1A: $N=12, K=4, L=2$	
$h_0(n)$	$g_0(n)$
−0.00178533012604	−0.00035706602521
0.01335887348208	−0.00018475350525
0.03609074349777	0.03259148575321
−0.03472219035063	0.01344990160212
0.04152506151211	−0.05846672525596
0.56035836869365	0.27464307660380
0.77458616704024	0.77956622415105
0.22752075128211	0.54097378940769
−0.16040926912642	−0.04031500786642
−0.06169425120853	−0.13320137936114
0.01709940838890	−0.00591212957013
0.00228522928787	0.01142614643933
Example 2: $N=12, K=3, L=3$	
$h_0(n)$	$g_0(n)$
−0.01558262447444	−0.00222608921063
−0.04943224834056	−0.04267917713309
0.21675411650608	0.02482915969003
0.74585008428240	0.49827824107483
0.61333711629573	0.79972651593977
−0.01550639700556	0.28678636149680
−0.12705042512607	−0.15642754715911
0.03236969097201	−0.03318989637197
0.01970114139115	0.04342764217365
−0.00619091208250	−0.00220469140539
−0.00005254340590	−0.00222290024716
0.00001656336077	0.00011594352537

is closer to zero for negative frequencies. At the same time, the Sobolev exponent decreases to 1.736. This is to be expected, as we have reduced the number of vanishing moments and at the same time increased the degree of approximation for the half-sample delay. The coefficients are given in Table II.

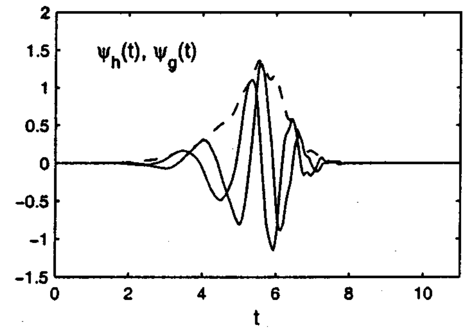

Fig. 2. Example 1B. Approximate Hilbert transform pair of orthonormal wavelet bases with $N = 12, K = 4, L = 2$, and minimum-phase spectral factorization.

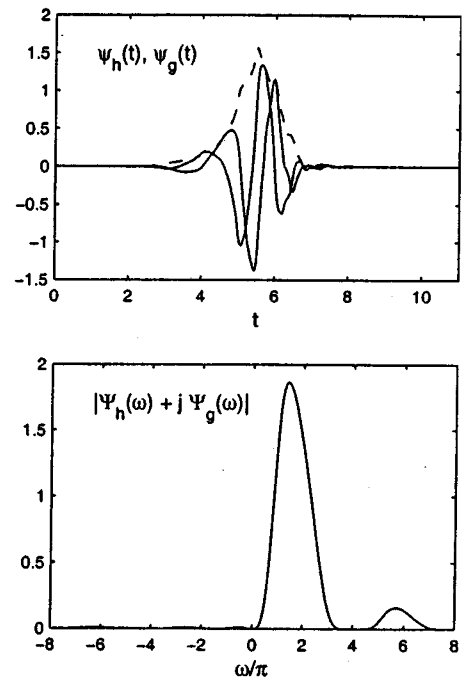

Fig. 3. Example 2. Approximate Hilbert transform pair of orthonormal wavelet bases with $N = 12, K = 3, L = 3,$, and mid-phase spectral factorization.

A. Directional 2-D Wavelets

One of the important applications of a Hilbert pair of wavelet bases is the implementation of directional 2-D (overcomplete) wavelet transforms, as illustrated in [7]. The directional wavelets are obtained by first defining a 2-D separable wavelet basis via

$$\psi_{h,1}(x,y) = \phi_h(x)\psi_h(y)$$
$$\psi_{h,2}(x,y) = \psi_h(x)\phi_h(y)$$
$$\psi_{h,3}(x,y) = \psi_h(x)\psi_h(y).$$

In addition, define $\psi_{g,i}$ similarly. Then, the six wavelets defined by

$$\psi_i(x,y) = \psi_{h,i}(x,y) + \psi_{g,i}(x,y) \quad (9)$$
$$\psi_{i+3}(x,y) = \psi_{h,i}(x,y) - \psi_{g,i}(x,y) \quad (10)$$

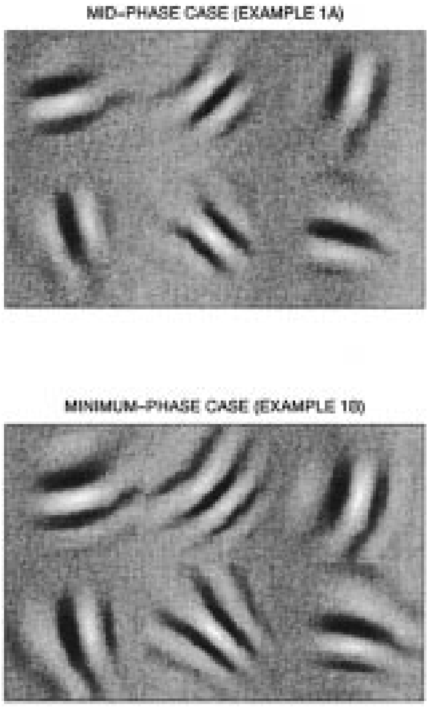

Fig. 4. Two-dimensional wavelets generated by an approximate Hilbert transform pair of 1-D wavelets, using a mid-phase spectral factor (Example 1A) and a minimum-phase spectral factor (Example 1B), respectively.

for $1 \le i \le 3$ are directional, as illustrated in Fig. 4. A wavelet transform based on these six wavelets can be implemented by taking the sum and difference of two separable 2-D DWTs. The resulting directional wavelet transform is two-times redundant. The inverse requires taking the sum and difference, dividing by 2, and the separable inverse DWTs. The four-times redundant DWT presented by Kingsbury is both directional and complex [8].

We note that the type of spectral factorization performed in the design procedure described above influences the quality of the directionality of the 2-D wavelets. For example, the wavelets of example 1A, which were constructed with a mid-phase spectral factor, lead to the six 2-D wavelets shown in the top panel of Fig. 4. On the other hand, the wavelets of example 1B, which were constructed with a minimum-phase spectral factor, lead to the six 2-D wavelets shown in the lower panel of Fig. 4. In this case, the directionality is not as clean as in the first case. Instead, some curvature is present. The figure shows that the mid-phase spectral factor can be preferable to the minimum-phase one for the implementation of directional wavelet transforms.

Evidently, better directional selectivity is obtained when the two wavelets have approximate (or exact) linear phase in addition to forming an approximate Hilbert transform pair. The procedure described above imposes a condition on the phase of of $\Psi_h(\omega)/\Psi_g(\omega)$, but it does not impose any condition of the phases of $\Psi_h(\omega)$ and $\Psi_g(\omega)$ individually. To ensure a high directional selectivity, one could impose directly that $H_0(\omega)$ and $G_0(\omega)$ have approximately linear phase rather than relying on the near linear-phase of a mid-phase spectral factor. For example, consider the system described in [9]. In [9], the filters $h_0(n)$ and $g_0(n)$ are related through a reversal

$$g_0(n) = h_0(N-1-n)$$

or equivalently

$$G_0(\omega) = e^{-j\omega(N-1)}\overline{H_0(\omega)}.$$

Combining with (4), we get the condition

$$H_0(\omega) = \overline{H_0(\omega)}e^{-j\omega\left((N-1)-\frac{1}{2}\right)}$$

or

$$\frac{H_0(\omega)}{\overline{H_0(\omega)}} = e^{-j\omega\left((N-1)-\frac{1}{2}\right)}.$$

Therefore

$$H_0(\omega) = |H_0(\omega)|e^{-j\omega\left(\frac{N-1}{2}-\frac{1}{4}\right)}. \tag{11}$$

In this case, $H_0(\omega)$ has linear phase, and the delay is offset by one quarter of a sample from the center of symmetry of a length N filter $h_0(n)$. As a CQF filter can not have exact linear phase, the filters given in [9] approximately satisfy (11). The design of orthonormal wavelets with approximate linear phase with non-integer delay has also been described in [13], [19], and [25]. An alternative, simple, way to obtain an approximate Hilbert transform pair of (biorthogonal) wavelets with approximate linear phase is to modify the spectral factorization approach, as described in the next section.

III. Biorthogonal Solutions

The quality of the directional 2-D wavelets derived from a pair of 1-D wavelets appears to depend on the wavelets having approximately linear phase, in addition to forming an approximate Hilbert transform pair. If biorthogonal wavelets are acceptable, then the procedure given in Section II can be modified to yield wavelet bases for which both $\Psi_h(\omega)$ and $\Psi_g(\omega)$ have approximately linear phase. They can then be used to generate 2-D wavelet bases with improved directional selectivity. To generate wavelets with approximate linear phase, we can follow the approach of [4] for the design of symmetric biorthogonal wavelets, in which the spectral factorization of a halfband filter is replaced by its factorization into two linear-phase, but different, filters. In the biorthogonal case, we denote the dual scaling functions and wavelets by $\tilde{\phi}_h(t), \tilde{\psi}_h(t), \tilde{\phi}_g(t)$, and $\tilde{\psi}_g(t)$. As we have a pair of biorthogonal wavelet bases, we have eight filters including the dual filters, corresponding to the filterbanks illustrated in Fig. 5.

The dual scaling function $\tilde{\phi}_h(t)$ and wavelet $\tilde{\psi}_h(t)$ are given by the equations

$$\tilde{\phi}_h(t) = \sqrt{2}\sum_n \tilde{h}_0(n)\tilde{\phi}_h(2t-n)$$
$$\tilde{\psi}_h(t) = \sqrt{2}\sum_n \tilde{h}_1(n)\tilde{\phi}_h(2t-n).$$

The dual scaling function $\tilde{\phi}_g$ and wavelet $\tilde{\psi}_g$ are similarly defined.

The goal will be to design the filters so that both the primary (synthesis) and dual (analysis) wavelets form approximate Hilbert transform pairs

$$\psi_g(t) = \mathcal{H}\{\psi_h(t)\} \quad \text{and} \quad \tilde{\psi}_g(t) = \mathcal{H}\{\tilde{\psi}_h(t)\}.$$

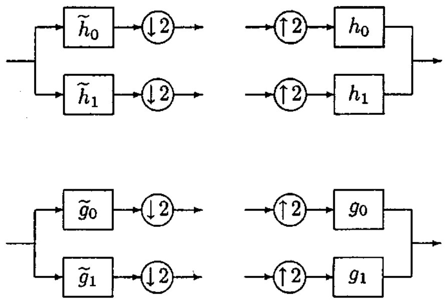

Fig. 5. Filterbank structures for the biorthogonal case.

If we define the product filters as

$$p_h(n) := \tilde{h}_0(n) * h_0(n)$$
$$p_g(n) := \tilde{g}_0(n) * g_0(n)$$

then the biorthogonality conditions can be written as

$$p_h(2n + n_o) = \delta(n) \tag{12}$$
$$p_g(2n + n_o) = \delta(n). \tag{13}$$

That is to say, p_h, p_g must both be halfband. With out loss of generality, we can assume that n_o is an odd integer. In that case, the highpass filters are given by the following expressions [4], [24]:

$$h_1(n) := (-1)^n \tilde{h}_0(n)$$
$$\tilde{h}_1(n) := -(-1)^n h_0(n)$$
$$g_1(n) := (-1)^n \tilde{g}_0(n)$$
$$\tilde{g}_1(n) := -(-1)^n g_0(n).$$

We will look for solutions of the form

$$h_0(n) = f(n) * d(n)$$
$$\tilde{h}_0(n) = \tilde{f}(n) * d(L - n)$$
$$g_0(n) = f(n) * d(L - n)$$
$$\tilde{g}_0(n) = \tilde{f}(n) * d(n).$$

The problem is to find $f(n)$ and $\tilde{f}(n)$ such that we have have the following.

1) The biorthogonality conditions (12) and (13) are satisfied.
2) The wavelets have K vanishing moments ($\tilde{K}$ vanishing moments for the dual wavelets).
3) The wavelets form an approximate Hilbert transform pair.

To obtain the half sample delay needed to ensure the approximate Hilbert property, we choose $d(n)$ to come from the flat delay allpass filter, as before:

$$A(z) = \frac{z^{-L} D(1/z)}{D(z)} \approx z^{-1/2} \quad \text{around } z = 1.$$

To ensure the vanishing moments properties, we take $F(z)$ and $\tilde{F}(z)$ to be of the form

$$F(z) = Q(z)(1 + z^{-1})^K$$
$$\tilde{F}(z) = \tilde{Q}(z)(1 + z^{-1})^{\tilde{K}}$$

where $K + \tilde{K}$ is odd. K denotes the number of vanishing moments of the primary (synthesis) wavelets, and $\tilde{K}$ denotes the number of vanishing moments of the dual (analysis) wavelets. The product filters p_h and p_g are then given by

$$P(z) := P_h(z) = P_g(z) \tag{14}$$
$$= Q(z)\tilde{Q}(z)(1 + z^{-1})^{K+\tilde{K}} D(z) D(1/z) z^{-L}. \tag{15}$$

To obtain the required halfband property, we find a symmetric odd-length sequence $r(n)$ so that

$$R(z)(1 + z^{-1})^{K+\tilde{K}} D(z) D(1/z) z^{-L}$$

is halfband. The symmetric sequence $r(n)$ can be obtained by solving a linear system of equations as in Section II. We can then obtain $Q(z)$ and $\tilde{Q}(z)$ by factoring $R(z)$

$$R(z) = Q(z)\tilde{Q}(z) \tag{16}$$

where both $q(n)$ and $\tilde{q}(n)$ are symmetric. As $q(n)$ and $\tilde{q}(n)$ are symmetric, so are $f(n)$ and $\tilde{f}(n)$. It follows that $h_0(n)$ and $g_0(n)$ are related by a reversal

$$g_0(n) = h_0(N - 1 - n) \tag{17}$$

and similarly

$$\tilde{g}_0(n) = \tilde{h}_0(N - 1 - n). \tag{18}$$

Therefore, $h_0, \tilde{h}_0, g_0$ and $\tilde{g}_0$ have approximately linear phase (because $d(n)$ does) in addition to satisfying (4) approximately. Note that the symmetric factorization (16) is not unique—many solutions are available. In addition, note that the sequences $f(n)$ and $\tilde{f}(n)$ do not need to have the same length.

Example 3: With $K = \tilde{K} = 4$ and $L = 2$, we can take the synthesis filters $h_0(n)$ and $g_0(n)$ to be of length $N = 13$. The analysis filters $\tilde{h}_0(n)$ and $\tilde{g}_0(n)$ will then be of length 11. Fig. 6 illustrates this solution obtained from a symmetric factorization. The plots of $|\Psi_h(\omega) + j\Psi_g(\omega)|$ and likewise $|\tilde{\Psi}_h(\omega) + j\tilde{\Psi}_g(\omega)|$ show that they approximate zero for $\omega < 0$, as expected. The coefficients $h_0(n)$ and $\tilde{h}_0(n)$ are given in Table III. The coefficients $g_0(n)$ and $\tilde{g}_0(n)$ are given by their reversed versions, as in (17) and (18). The Sobolev exponent of $\tilde{\psi}_h(t)$ and $\tilde{\psi}_g(t)$ is 1.731, whereas the Sobolev exponent of $\psi_h(t)$ and $\psi_g(t)$ is 2.232.

Fig. 7 illustrates the analysis and synthesis directional 2-D wavelets derived from the 1-D wavelets using (9) and (10).

IV. IIR Solutions

The spectral factorization approach can also be used to construct orthogonal wavelet bases based on recursive infinite impulse response (IIR) digital filters, where $H_0(z)$ is a rational function of z. Wavelets based on rational scaling filters have been discussed, for example, in [6], [15], [17], and [21]. IIR filters often require lower computational complexity than finite impulse response (FIR) filters. Analogous to the approach given in [6], but with the flat delay filter included, approximate Hilbert

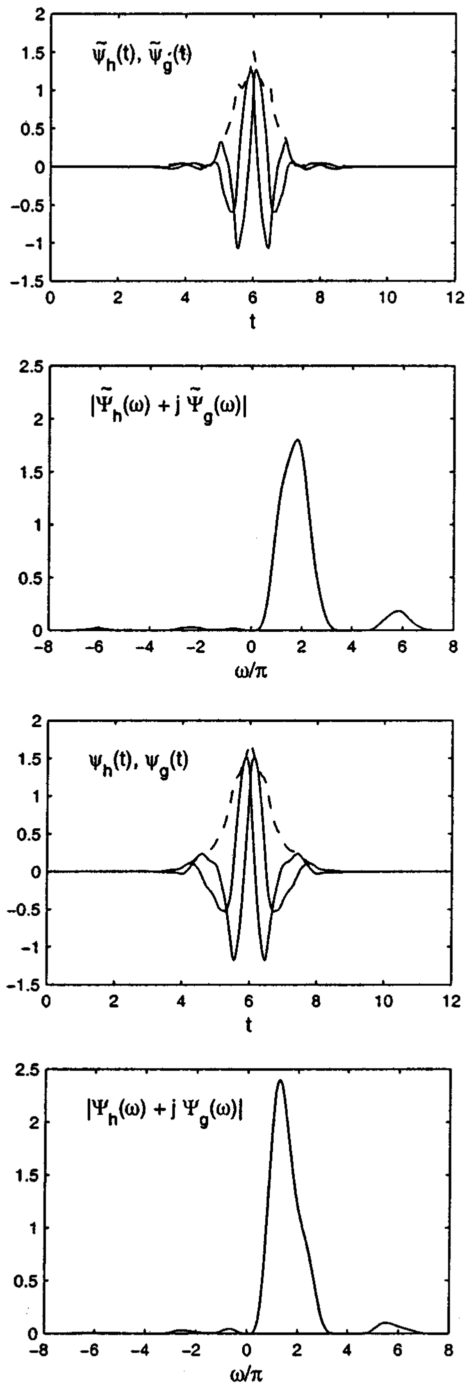

Fig. 6. Example 3: Approximate Hilbert transform pair of biorthogonal wavelet bases with $N = 13, K = \tilde{K} = 4, L = 2$.

transform pairs of IIR wavelets can be obtained with the following form:

$$H_0(z) = \frac{(1+z^{-1})^K D(z)}{C(z^2)}$$
$$H_1(z) = \frac{(1-z^{-1})^K D(-1/z) z^{-L}}{C(z^2)}$$
$$G_0(z) = \frac{(1+z^{-1})^K D(1/z) z^{-L}}{C(z^2)}$$
$$G_1(z) = \frac{(1-z^{-1})^K D(-z)}{C(z^2)}.$$

TABLE III
COEFFICIENTS FOR EXAMPLE 3

Example 3: $N = 13, K = \tilde{K} = 4, L = 2$	
$\tilde{h}_0(n)$	$h_0(n)$
0	−0.00030453648331
0.01339704408541	−0.00015432782903
−0.00678912656592	0.02776733337848
−0.15762026783144	0.01114383888330
−0.04891448800028	−0.04715783038404
0.64995392445932	0.23730846518842
0.93376892975667	0.65840893759976
0.27109270647926	0.47625050860818
−0.19103597922739	0.03941149716348
−0.07239603482308	−0.02885152385159
0.02007744522348	0.03050406232874
0.00267940881708	0.01140982018728
0	−0.00152268241655

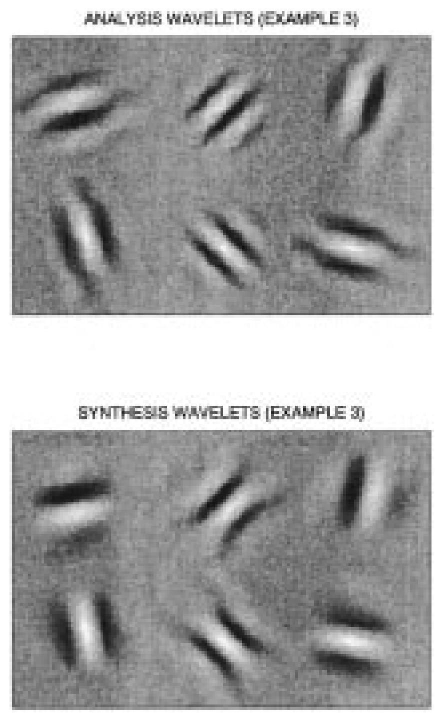

Fig. 7. Two-dimensional wavelets generated by an approximate Hilbert transform pair of biorthogonal 1-D wavelets (Example 3).

The product filter is given by

$$P(z) := H_0(z)H_0(1/z) = G_0(z)G_0(1/z) = \frac{(z+2+z^{-1})^K D(z)D(1/z)}{C(z^2)\,C(1/z^2)}.$$

Defining

$$V(z) := (z+2+z^{-1})^K D(z)D(1/z)$$

the orthogonality condition $P(z) + P(-z) = 2$ can be written as

$$\frac{V(z)+V(-z)}{2} = C(z^2)C(1/z^2). \qquad (19)$$

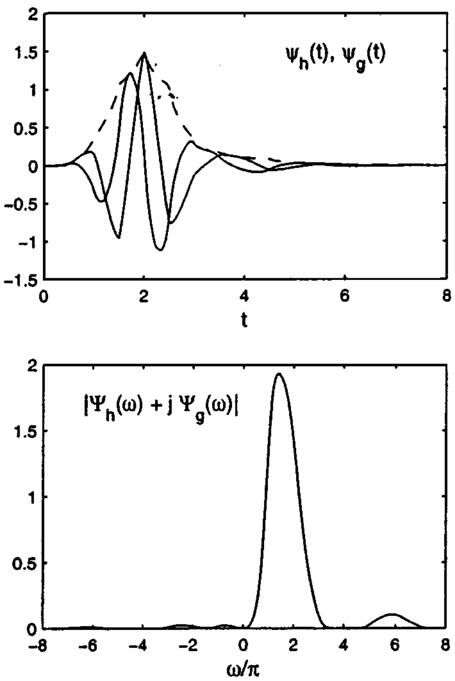

Fig. 8. Example 4. Approximate Hilbert transform pair of orthonormal IIR wavelet bases with $K = 2, L = 2$.

$C(z)$ can be found by spectral factorization; note that the left-hand side of (19) is a function of z^2. A stable filter is obtained using the minimum-phase spectral factor in (19).

Example 4: With two vanishing moments $(K = 2)$ and $L = 2$, we obtain the following stable causal transfer functions

$$H_0(z) = \frac{1 + 4z^{-1} + 5.2z^{-2} + 2.4z^{-3} + 0.2z^{-4}}{6.6495 + 2.3714z^{-2} + 0.0301z^{-4}}$$
$$G_0(z) = \frac{0.2 + 2.4z^{-1} + 5.2z^{-2} + 4z^{-3} + z^{-4}}{6.6495 + 2.3714z^{-2} + 0.0301z^{-4}}.$$

Fig. 8 illustrates the two wavelets $\psi_h(t), \psi_g(t)$ and the magnitude of $\Psi_h(\omega) + j\Psi_g(\omega)$.

V. Comparison to Kingsbury's Filters

It is interesting to compare the filters obtained here with those presented by Kingsbury in [7]–[10], where the complex dual-tree DWT is introduced. In [7] and [8], Kingsbury presents linear-phase biorthogonal solutions where one scaling function is symmetric about an integer, whereas the other scaling function is symmetric about an integer plus 0.5. In this case, the (approximate) half-sample delay [see (4)] can be achieved by making the frequency responses magnitudes (approximately) equal $|H_0(\omega)| \approx |G_0(\omega)|$. The family of filter banks introduced in [9] and [10] are orthogonal but with the property that one scaling function is the time-reversed version of the other. Then, the frequency responses magnitudes are equal, and it remains to design the phase response of the filter to achieve (approximately) the half-sample shift [see (11)]. On the other hand, the approach described in this paper does not impose any condition on the phase-response of the filters. The orthogonal solutions given in this paper do not have approximately linear phase (for short filters the mid-phase spectral-factorization does not always provide a solution with a nearly linear phase response). However, the biorthogonal solutions given in this paper have approximately linear phase. As a result, the orthogonal solutions by Kingsbury are more nearly linear phase than the orthogonal solutions given here; likewise, the linear-phase biorthogonal solutions by Kingsbury are more nearly orthogonal than the approximately linear-phase biorthogonal solutions given here.

In addition, the design procedure developed in this paper and the ones described by Kingsbury are quite different. Kingsbury's design procedures are based on minimizing the aliasing of a filterbank after it is iterated a fixed number stages. The minimization is performed over a parameterized space of perfect reconstruction filterbanks using iterative optimization algorithms. On the other hand, the approach taken in this paper is based on the limit functions [defined by (2) and (3)] and on vanishing moments. As a result, the limit functions $\psi_h(t)$ and $\psi_g(t)$ are closer to forming a Hilbert pair for the filters given in this paper than those presented by Kingsbury, whereas the solutions by Kingsbury do this better for the first several stages.

VI. Conclusion

This paper presents simple procedures for the design of pairs of wavelet bases where the two wavelets form an approximate Hilbert transform pair. The approach proposed here is analogous to the Daubechies construction of compactly supported wavelets with vanishing moments but where the approximate Hilbert transform relation is added by way of incorporating a flat delay filter.

The approach is based on a characterization of Hilbert transform pairs of wavelets bases given in [18]. The formulation of the problem, using a flat delay allpass filter, makes it possible to employ the spectral factorization design method as introduced in [22] for the design of CQF filters. Note that even though an allpass filter arises in the problem formulation, the filters we obtain are FIR (IIR solutions are also available, as described in Section IV).

Given an allpass filter, the proposed design method produces short filters with a specified number of vanishing wavelet moments. Although a flat delay filter was used here, any other allpass filters that approximate a delay of a half sample could be used instead. The degree of the allpass filter controls the degree to which the half-sample offset property is satisfied. A Matlab program for the procedure is given, and examples are also given to illustrate the results. Programs are available on the Web at http://taco.poly.edu/selesi.

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