Hilbert Transform Pairs of Wavelet Bases

Ivan W. Selesnick, *Member, IEEE*

***Abstract*—This paper considers the design of pairs of wavelet bases where the wavelets form a Hilbert transform pair. The derivation is based on the limit functions defined by the infinite product formula. It is found that the scaling filters should be offset from one another by a half sample. This gives an alternative derivation and explanation for the result by Kingsbury, that the dual-tree DWT is (nearly) shift-invariant when the scaling filters satisfy the same offset.**

I. Introduction

Several authors have proposed signal processing methods that call for two wavelet transforms, where one wavelet is (approximately) the Hilbert transform of the other. For example, Abry and Flandrin suggested it for transient detection [2] and turbulence analysis [1], Kingsbury suggested it for the complex dual-tree discrete wavelet transform (DWT) [7], [8], and Ozturk *et al.* suggested it for waveform encoding [9]. In addition, Freeman and Adelson employ the Hilbert transform in the development of steerable filter banks [5]. Also of related interest is the paper by Beylkin and Torrésani [3].

The lowpass filters $h_0(n)$, $g_0(n)$ fully determine the two orthogonal wavelet bases. But how can we choose h_0 and g_0 so that the two wavelets they generate will form a Hilbert transform pair? This is the question addressed in this paper.

Kingsbury found that the dual-tree DWT is nearly shift-invariant when the lowpass filters of one DWT interpolate midway between the lowpass filters of the second DWT. This paper considers the limit functions defined by the infinite-product formula, rather than the (near) shift-invariance of a finitely iterated filter bank as in [7], and arrives at the same condition. This letter thereby gives an alternative explanation for why the scaling filters should be designed to be offset from each other by a half sample delay.

A. Preliminaries

Let the filters $h_0(n)$, $h_1(n)$ represent a conjugate quadrature filter (CQF) pair. That is

$$\sum_n h_0(n)h_0(n+2k) = \delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

and $h_1(n) = (-1)^{(1-n)} h_0(1-n)$. Equivalently, in terms of the Z-transform, we have

$$H_0^z(z)H_0^z(1/z) + H_0^z(-z)H_0^z(-1/z) = 2$$

and

$$H_1^z(z) = \frac{1}{z} H_0^z(-1/z).$$

We use the notation $H^z(z)$ for the Z-transform of $h(n)$. Then the frequency response of the filter is $H(\omega) = H^z(e^{j\omega})$. The filters $g_0(n)$, $g_1(n)$ represent another CQF pair. In this letter, we assume $h_i(n)$, $g_i(n)$ are real-valued filters. The dilation and wavelet equations give the scaling and wavelet functions

$$\phi_h(t) = \sqrt{2} \sum_n h_0(n)\phi_h(2t-n)$$

$$\psi_h(t) = \sqrt{2} \sum_n h_1(n)\phi_h(2t-n).$$

The scaling function $\phi_g(t)$ and wavelet $\psi_g(t)$ are defined similarly but with filters $g_0(n)$ and $g_1(n)$.

II. Hilbert Transform Pairs

Recall the definition of the Hilbert transform. $\psi_g(t)$ is the Hilbert transform of $\psi_h(t)$ if

$$\Psi_g(\omega) = \begin{cases} -j\Psi_h(\omega), & \omega > 0 \\ j\Psi_h(\omega), & \omega < 0. \end{cases} \tag{1}$$

Suppose the two lowpass filters are related as follows:

$$G_0(\omega) = H_0(\omega)e^{-j\theta(\omega)}$$

where $\theta(\omega)$ is 2π-periodic. We will see how to choose the phase $\theta(\omega)$ so that the two wavelets generated by h_0 and g_0 form a Hilbert transform pair. We proceed by considering three questions.

A. How is $\phi_g(t)$ Related to $\phi_h(t)$?

By the infinite-product formula, we have

$$\Phi_h(\omega) = \mathcal{F}\{\phi_h(t)\} = \Phi_h(0) \prod_{k=1}^{\infty} \left\{ \frac{1}{\sqrt{2}} H_0\left(\frac{\omega}{2^k}\right) \right\}.$$

Similarly for $\Phi_g(\omega)$

$$\begin{aligned}
\Phi_g(\omega) &= \mathcal{F}\{\phi_g(t)\} \\
&= \Phi_g(0) \prod_{k=1}^{\infty} \left\{ \frac{1}{\sqrt{2}} G_0\left(\frac{\omega}{2^k}\right) \right\} \\
&= \Phi_g(0) \prod_{k=1}^{\infty} \left\{ \frac{1}{\sqrt{2}} H_0\left(\frac{\omega}{2^k}\right) e^{-j\theta(\omega/2^k)} \right\} \\
&= \Phi_g(0) \prod_{k=1}^{\infty} \left\{ \frac{1}{\sqrt{2}} H_0\left(\frac{\omega}{2^k}\right) \right\} \\
&\quad \cdot \exp\left[-j \sum_{k=1}^{\infty} \theta(\omega/2^k) \right] \\
&= \frac{\Phi_g(0)}{\Phi_h(0)} \Phi_h(\omega) \\
&\quad \cdot \exp\left[-j \sum_{k=1}^{\infty} \theta(\omega/2^k) \right].
\end{aligned}$$

Manuscript received October 31, 2000. This work was supported by the National Science Foundation CAREER grant CCR-987452. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. A. M. Sayeed.

The author is with the Electrical Engineering Department, Polytechnic University, Brooklyn, NY 11201 USA.

Publisher Item Identifier S 1070-9908(01)03294-1.

Having $\Phi_g(0) = \Phi_h(0) = 1$ for orthogonal wavelet bases gives

$$\Phi_g(\omega) = \Phi_h(\omega) e^{-j\Sigma_{k=1}^{\infty}\theta(\omega/2^k)}. \tag{2}$$

B. How is $G_1(\omega)$ Related to $H_1(\omega)$?

The CQF filter bank has

$$H_1^z(z) = \frac{1}{z} H_0^z(-1/z) \quad \text{or} \quad H_1(\omega) = e^{-j\omega}\overline{H_0(\omega - \pi)}$$

where the overbar denotes complex conjugation. Similarly

$$\begin{aligned} G_1(\omega) &= e^{-j\omega}\overline{G_0(\omega - \pi)} \\ &= e^{-j\omega}\overline{H_0(\omega - \pi)e^{-j\,\theta(\omega-\pi)}} \\ &= e^{-j\omega}\overline{H_0(\omega - \pi)}e^{j\,\theta(\omega-\pi)} \\ &= H_1(\omega)e^{j\,\theta(\omega-\pi)}. \end{aligned}$$

C. How is $\psi_g(t)$ Related to $\psi_h(t)$?

The Fourier transform (FT) of $\psi_h(t)$ is given by

$$\Psi_h(\omega) = \mathcal{F}\{\psi_h(t)\} = \frac{1}{\sqrt{2}} H_1\left(\frac{\omega}{2}\right)\Phi_h\left(\frac{\omega}{2}\right)$$

and similarly for $\Psi_g(\omega)$

$$\begin{aligned} \Psi_g(\omega) &= \mathcal{F}\{\psi_g(t)\} \\ &= \frac{1}{\sqrt{2}} G_1\left(\frac{\omega}{2}\right)\Phi_g\left(\frac{\omega}{2}\right) \\ &= \frac{1}{\sqrt{2}} H_1\left(\frac{\omega}{2}\right) e^{j\theta(\omega/2-\pi)}\Phi_g\left(\frac{\omega}{2}\right) \\ &= \frac{1}{\sqrt{2}} H_1\left(\frac{\omega}{2}\right) e^{j\theta(\omega/2-\pi)}\Phi_h\left(\frac{\omega}{2}\right) \\ &\quad \cdot \exp\left[-j\sum_{k=1}^{\infty}\theta(\omega/2^{k+1})\right] \\ &= \frac{1}{\sqrt{2}} H_1\left(\frac{\omega}{2}\right)\Phi_h\left(\frac{\omega}{2}\right) \\ &\quad \cdot \exp\left[j\theta(\omega/2-\pi) - j\sum_{k=1}^{\infty}\theta(\omega/2^{k+1})\right] \\ &= \Psi_h(\omega)\exp\left\{j\left[\theta(\omega/2-\pi) - \sum_{k=1}^{\infty}\theta(\omega/2^{k+1})\right]\right\}. \end{aligned}$$

Therefore, we can write

$$\Psi_g(\omega) = \Psi_h(\omega)\exp\left\{j\left[\theta(\omega/2-\pi) - \sum_{k=2}^{\infty}\theta(\omega/2^{k})\right]\right\}. \tag{3}$$

D. Phase Condition

Can we choose $\theta(\omega)$ so that $\psi_h(t)$, $\psi_g(t)$ make a Hilbert transform pair? From (3) and (1), we see that $\theta(\omega)$ must satisfy the following condition:

$$\theta(\omega/2-\pi) - \sum_{k=2}^{\infty}\theta(\omega/2^k) = \begin{cases} -\dfrac{\pi}{2}, & \omega > 0 \\ \dfrac{\pi}{2}, & \omega < 0. \end{cases} \tag{4}$$

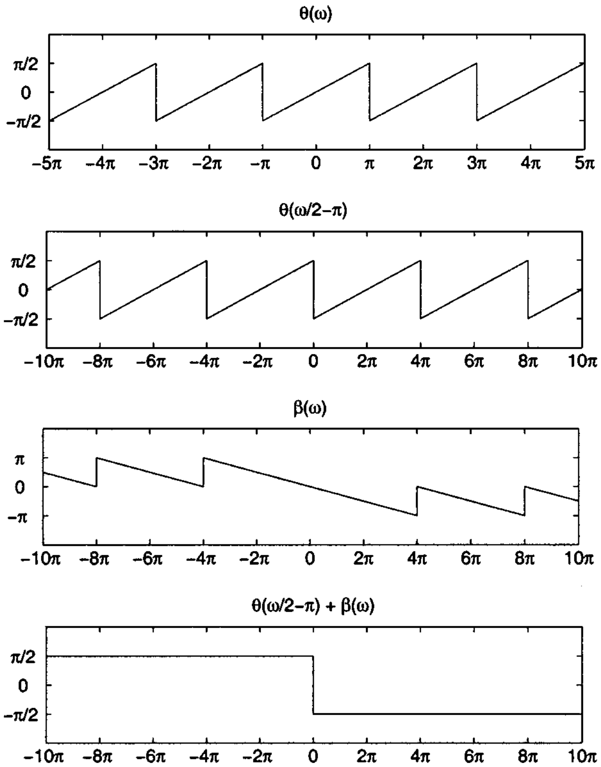

Fig. 1. Phase functions arising in the derivation of (5).

We now show that if the 2π-periodic function $\theta(\omega)$ is defined as

$$\theta(\omega) = \frac{\omega}{2}, \qquad |\omega| < \pi \tag{5}$$

as illustrated in Fig. 1, then condition (4) holds and $\psi_h(t)$, $\psi_g(t)$ make a Hilbert transform pair. First, note that if $\theta(\omega)$ is given by (5), then $\theta(\omega/2-\pi)$ is a 4π-periodic function given by

$$\theta(\omega/2-\pi) = \begin{cases} -\dfrac{\pi}{2} + \dfrac{\omega}{4}, & 0 < \omega < 2\pi \\ \dfrac{\pi}{2} + \dfrac{\omega}{4}, & -2\pi < \omega < 0 \end{cases}$$

as illustrated in Fig. 1. Now if we call the second term in (4) $\beta(\omega)$

$$\beta(\omega) = -\sum_{k=2}^{\infty}\theta(\omega/2^k)$$

and if $\theta(\omega)$ is given by (5), then we can show that $\beta(\omega)$ is given by

$$\beta(\omega) = \begin{cases} -\dfrac{\omega}{4}, & |\omega| < 4\pi \\ \beta(\omega - 4\pi), & 4\pi < \omega \\ \beta(\omega + 4\pi), & \omega < -4\pi \end{cases}$$

as illustrated in Fig. 1. Adding $\theta(\omega/2-\pi)$ and $\beta(\omega)$, we get the graph shown in Fig. 1. Evidently, one finds that condition (4) is indeed satisfied by the choice (5).

1) Theorem: If $H_0(\omega)$ and $G_0(\omega)$ are lowpass CQF filters (scaling filters) with

$$G_0(\omega) = H_0(\omega)e^{-j(\omega/2)} \quad \text{for } |\omega| < \pi$$

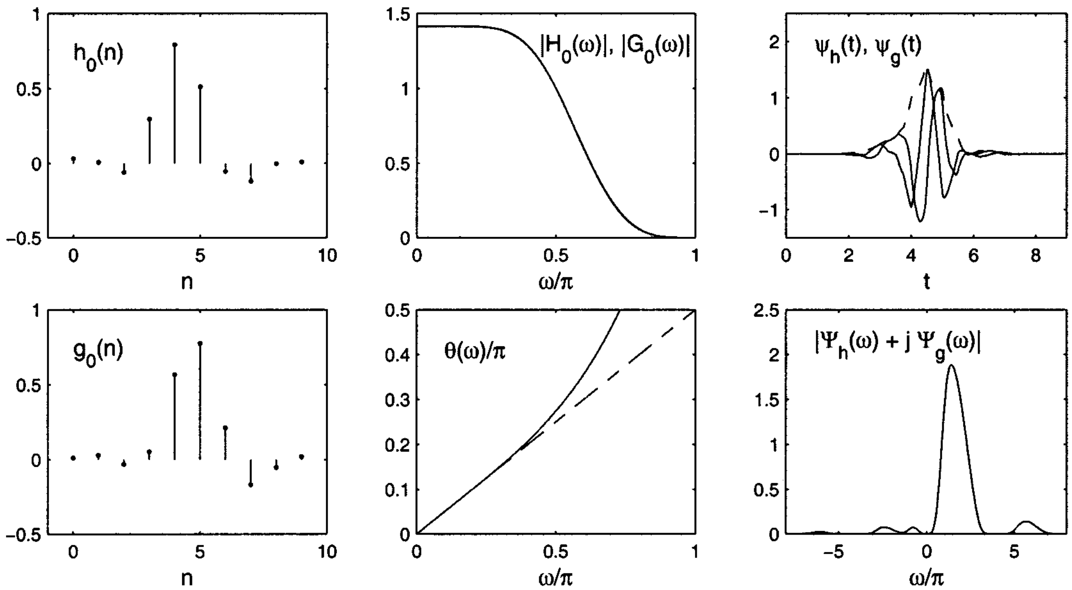

Fig. 2. Example 1: Approximate Hilbert transform pair of orthonormal wavelet bases, with $N = 10$, $K = 4$, $L = 5$.

then the corresponding wavelets are a Hilbert transform pair

$$\psi_g(t) = \mathcal{H}\{\psi_h(t)\}.$$

Equivalently, the digital filter $g_0(n)$ is a *half-sample* delayed version of $h_0(n)$

$$g_0(n) = h_0(n - 1/2).$$

As a half-sample delay can not be implemented with a finite impulse response (FIR) filter (not even a rational IIR filter can be exact), it is necessary to make an approximation.

III. Design Problem

The phase condition leads to following design problem: construct the shortest FIR filters h_0, g_0, such that they possess a specified number of zero moments, and that

$$G_0(\omega) \approx H_0(\omega)e^{-j(\omega/2)}.$$

The error function is given by

$$E_1(\omega) = G_0(\omega) - H_0(\omega)e^{-j(\omega/2)}.$$

If we define a new function $E_2(\omega)$ by

$$E_2(\omega) := E_1(2\omega) = G_0(2\omega) - H_0(2\omega)e^{-j\omega}$$

then the Z-transform $E_2^z(z)$ is a polynomial

$$E_2^z(z) = G_0^z(z^2) - H_0^z(z^2)z^{-1}.$$

Let us choose $z = 1$ as the point of approximation. To make $E_2^z(z)$ close to zero at $z = 1$, we can ask that

$$G_0^z(z^2) - H_0^z(z^2)z^{-1} = Q_e(z)(1 - z^{-1})^L$$

TABLE I
Filter Coefficients for the Examples

Example 1: $N = 10, K = 4, L = 5$	
$h_0(n)$	$g_0(n)$
0.03221257407420	0.01123179664593
0.00820937853576	0.02938462747110
−0.06023981115681	−0.02941520794631
0.29737132183851	0.05228807988494
0.79149943086392	0.56614863255854
0.51279103306800	0.77383926442488
−0.05414137333876	0.21123282805692
−0.11999180398584	−0.16831623187690
−0.00222403925601	−0.05209126812854
0.00872685173012	0.01991104128252
Example 2: $N = 10, K = 3, L = 7$	
$h_0(n)$	$g_0(n)$
0.00419528584157	0.00051763584333
−0.03976408134143	−0.00016716564000
−0.08807084231507	−0.09187942035452
0.28789890325798	0.02408482114448
0.80289644768232	0.61837942541527
0.50734324828341	0.75480699639212
−0.04438514804476	0.17400853530401
−0.05179712664076	−0.09044673462008
0.03247103802248	0.00608060497845
0.00342583762736	0.01882886391002

where L represents the degree of approximation to the half-sample delay. If K is the number of zero wavelet moments, and L is the parameter for controlling the half-sample delay approximation, then we have the following design equations, which we wish to solve for the filters h_0 and g_0 of minimal length:

1) $\Sigma_n\ h_0(n)h_0(n + 2k) = \delta(k)$;
2) $\Sigma_n\ g_0(n)g_0(n + 2k) = \delta(k)$;
3) $H_0^z(z) = Q_h(z)(1 + z^{-1})^K$;

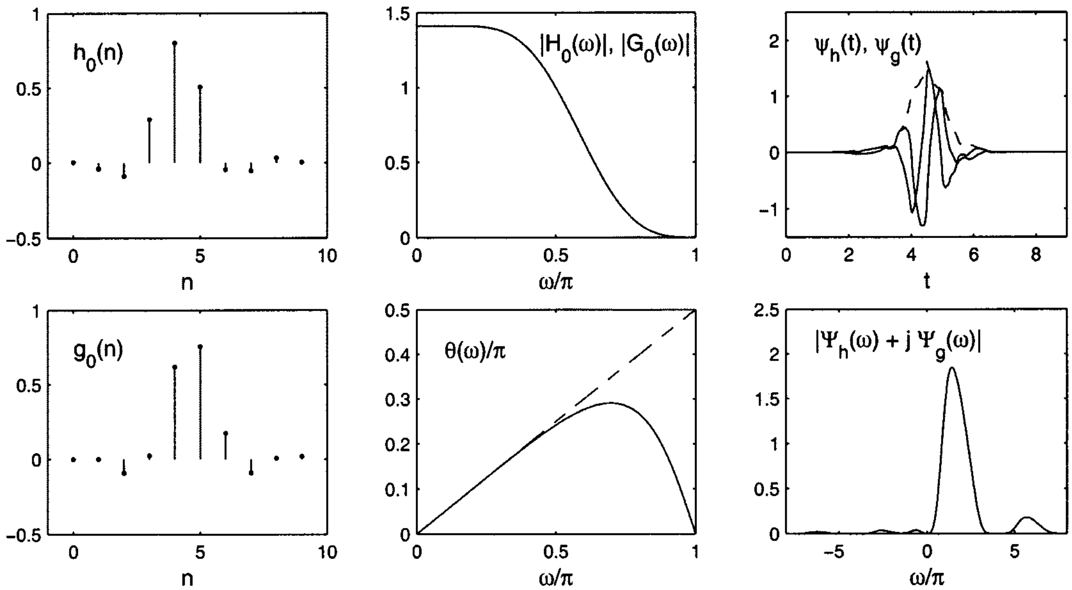

Fig. 3. Example 2. Approximate Hilbert transform pair of orthonormal wavelet bases, with $N = 10$, $K = 3$, $L = 7$.

4) $G_0^z(z) = Q_g(z)(1 + z^{-1})^K$;
5) $G_0^z(z^2) - z^{-1}H_0^z(z^2) = Q_e(z)(1 - z^{-1})^L$

We illustrate two examples obtained using this design problem. The design equations are nonlinear. However, solutions can be obtained using Gröbner bases [4]. We used the software *Singular* [6] to obtain the Gröbner needed for the following examples.

Example 1: With $K = 4$ and $L = 5$, we find that the shortest filters $h_0(n)$ and $g_0(n)$ satisfying the conditions are of length 10. Fig. 2 illustrates one of the several solutions that exist. Note that $|H_0(\omega)|$ and $|G_0(\omega)|$ are indistinguishable in the plot. The plot of $\theta(\omega)$ shows its agreement with $\omega/2$ near $\omega = 0$. The plot of the function $|\Psi_h(\omega) + j\,\Psi_g(\omega)|$ shows that it approximates zero for $\omega < 0$ as expected if ψ_h and ψ_g make a Hilbert transform pair. Table I tabulates the coefficients.

Example 2: With $K = 3$ and $L = 7$, the minimal lengths of $h_0(n)$ and $g_0(n)$ is again ten samples. Fig. 3 illustrates one of the several solutions. It can be seen that the wavelets are not quite as smooth as in the previous example, but that $|\Psi_h(\omega) + j\,\Psi_g(\omega)|$ is closer to zero for negative frequencies. This is to be expected, as we have reduced the number of zero moments and at the same time improve the half-sample delay approximation.

Using longer filters, we have obtained solutions that have both good smoothness and good half-sample delay properties. Note that $h_0(n)$ and $g_0(n)$ do not need to have (near) linear phase in order for $\psi_h(t)$ and $\psi_g(t)$ to make a Hilbert transform pair, although it may be desirable for other reasons depending on the application.

IV. Conclusion

Using the infinite product formula, it was shown that for two orthogonal wavelets to form a Hilbert transform pair, the scaling filters should be offset by a half sample. An example was presented to illustrate the trade-off between the number of zero wavelet moments and the degree of half-sample delay approximations.

References

[1] P. Abry, *Ondelettes et Turbulences*. Paris, France: Diderot, 1997.
[2] P. Abry and P. Flandrin, "Multiresolution transient detection," in *Proc. IEEE Signal Processing Int. Symp. Time-Frequency and Time-Scale Analysis*, Philadelphia, PA, Oct. 1994, pp. 225–228.
[3] G. Beylkin and B. Torrésani, "Implementation of operators via filter banks: Autocorrelation shell and hardy wavelets," *Appl. Comput. Harmon. Anal.*, vol. 3, pp. 164–185, 1996.
[4] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algegraic Geometry and Commutative Algebra*. Berlin, Germany: Springer-Verlag, 1991.
[5] W. T. Freeman and E. H. Adelson, "The design and use of steerable filters," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 13, pp. 891–906, Sept. 1991.
[6] G.-M. Greuel, G. Pfister, and H. Schönemann, "Singular reference manual," Center Computer Algebra, Univ. Kaiserslautern, Germany, http://www.singular.uni-kl.de/, May 1997.
[7] N. G. Kingsbury, "Image processing with complex wavelets," *Phil. Trans. R. Soc. London A*, Sept. 1999.
[8] ——, "A dual-tree complex wavelet transform with improved orthogonality and symmetry properties," in *Proc. IEEE Int. Conf. Image Processing*, Vancouver, BC, Canada, Sept. 10–13, 2000.
[9] E. Ozturk, O. Kucur, and G. Atkin, "Waveform encoding of binary signals using a wavelet and its Hilbert transform," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing (ICASSP)*, Istanbul, Turkey, June 5–9, 2000.