# Multiwavelet Bases with Extra Approximation Properties

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***Abstract*—This paper highlights the differences between traditional wavelet and multiwavelet bases with equal approximation order. Because multiwavelet bases normally lack important properties that traditional wavelet bases (of equal approximation order) possess, the associated discrete multiwavelet transform is less useful for signal processing unless it is preceded by a preprocessing step (prefiltering). This paper examines the properties and design of orthogonal multiwavelet bases, with approximation order >1 that possess those properties that are normally absent. For these "balanced" bases (so named by Lebrun and Vetterli), prefiltering can be avoided. By reorganizing the multiwavelet filter bank, the development in this paper draws from results regarding the approximation order of $M$-band wavelet bases. The main result thereby obtained is a characterization of balanced multiwavelet bases in terms of the divisibility of certain transfer functions by powers of ($z^{-2r} - 1)/(z^{-1} - 1$). For traditional wavelets ($r = 1$), this specializes to the usual factor ($z + 1)^K$.**

***Index Terms*—Multirate filter banks, multiwavelet transforms, prefilters, time-varying filter banks, wavelet transforms.**

## I. Introduction

MULTIWAVELET bases (wavelet bases based on multiple scaling functions) have been investigated for several years now. With this generalization, it is possible to construct orthogonal (real-valued) bases for which the scaling functions have compact support, approximation order greater than 1, and symmetry, which is not possible with traditional wavelet bases. This is demonstrated by the example given by Donovan *et al.* [7], [9]. The important characterization of multiwavelets with specified approximation order, in terms of scaling coefficients, which is significantly more complicated than is so for traditional wavelet bases (those based on a single scaling function), has been developed and described in [2], [7], [12], [13], [19], [20], [22], and [28].

The differences between traditional wavelet and multiwavelet bases, with equal approximation order (zero moment properties), are highlighted in this paper. Traditional wavelet bases have several attributes that multiwavelet bases, of equal approximation order, do not normally possess. Specifically, the bandpass channels of the iterated filter bank associated with a multiwavelet basis of approximation order $K$ do not necessarily annihilate discrete-time polynomials of degree $k < K$. In other words, for multiwavelets, zero moments of the wavelet $\psi_i(t)$ on $R$ do not necessarily imply zero moments of the wavelet filter $h_i(n)$ on $Z$. Because multiwavelet bases with approximation order $K$ lack those annihilation properties, a preprocessing step (prefiltering) is fundamentally more important for the discrete multiwavelet transform (DMWT) than is so for the traditional DWT. Accordingly, several authors have addressed the design of prefilters specifically suited for the DMWT [11], [24], [35], [36]. On the other hand, it is also natural to investigate the design of multiwavelets, with specified approximation order, that *do* possess those attributes that are normally absent. Lebrun and Vetterli addressed this topic and, in [16] and [17], coined the term "balanced" multiwavelets.

In this paper, we also examine such multiwavelets and characterize them in terms of the associated scaling filter coefficients. It is found that for an orthogonal multiwavelet basis to be balanced, certain transfer functions must be divisible by powers of $(z^{-2r} - 1)/(z^{-1} - 1)$. That is, certain transfer functions must have zeros at roots of unity, excepting $z = 1$. The same divisibility condition arises in the case of $M$-band wavelet bases [14], [26]. In fact, the development in this paper uses a reorganization of the multiwavelet (vector) filter bank as a multichannel scalar filter bank. In that way, the development draws from results regarding the approximation order of $M$-band wavelet bases.

The success of wavelet bases in applications often depends in part on the short support of the scaling function in addition to its approximation order. Both support and approximation order are important; however, there is a fundamental tradeoff between these competing design criteria. For traditional wavelet bases, the length of the scaling function is the only parameter with which we can manage that tradeoff. The generalization provided by bases with multiple scaling functions permits greater flexibility in managing this tradeoff. In particular, the time localization of one scaling function can be improved if that of other scaling function(s) are relaxed. A balanced multiwavelet basis where some scaling functions have improved time localization behavior may be advantageous for some applications, e.g., denoising via wavelet coefficient thresholding. This paper gives examples of balanced orthogonal multiwavelet bases with approximation orders 2 and 3 for which the scaling functions are of differing support.

Section II gives preliminary notation and discusses the filter bank associated with a multiwavelet basis and its reorganization into a scalar filter bank. Section III outlines the differences in the meaning of approximation order for scalar

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and multiwavelets and discusses the use of prefilters for overcoming the lack of the annihilation properties of (unbalanced) multiwavelets. Section IV introduces a characterization of balanced multiwavelets. With this characterization, Section IV designs examples of orthogonal multiwavelets balanced up to their approximation order $K > 1$ that are distinct from Daubechies' wavelets.

## II. PRELIMINARIES

A multiwavelet basis (of multiplicity $r$) is characterized by $r$ scaling functions and $r$ wavelet functions. The scaling space $\mathcal{V}_j$ is defined by

$$\mathcal{V}_j = \operatorname*{Span}_k \{\phi_0(2^j t - k), \cdots, \phi_{r-1}(2^j t - k)\} \tag{1}$$

and the wavelet space $\mathcal{W}_j$ is defined by

$$\mathcal{W}_j = \operatorname*{Span}_k \{\psi_0(2^j t - k), \cdots, \psi_{r-1}(2^j t - k)\}. \tag{2}$$

For wavelet bases based on a single scaling function (traditional wavelet bases), from the nesting condition $\mathcal{V}_j \subset \mathcal{V}_{j+1}$, the dilation equation is obtained. The same is true for multiwavelets, where

$$\underline{\phi}(t) = \sqrt{2} \sum_n C_0(n)\, \underline{\phi}(2t - n) \tag{3}$$

is the matrix dilation equation (or refinement equation), where

$$\underline{\phi}(t) = \begin{pmatrix} \phi_0(t) \\ \phi_1(t) \end{pmatrix} \tag{4}$$

for a multiplicity $r = 2$ multiwavelet basis. In (3), $C_0(n)$ are $r \times r$ matrices. The sequence $C_0(n)$ is the matrix scaling filter, which is an $r$-input, $r$-output filter. The notation for $C_0(n)$ used in this paper is

$$[C_0(n)]_{i,j} = h_i(nr + j) \tag{5}$$

for example, when $r = 2$

$$C_0(0) = \begin{pmatrix} h_0(0) & h_0(1) \\ h_1(0) & h_1(1) \end{pmatrix}$$
$$C_0(1) = \begin{pmatrix} h_0(2) & h_0(3) \\ h_1(2) & h_1(3) \end{pmatrix}, \text{ etc.} \tag{6}$$

In this case, there are two scaling filters $h_0(n)$ and $h_1(n)$. In general, there are $r$ scaling filters and $r$ wavelet filters. The transfer functions $H_i(z)$ are given by

$$H_i(z) = \sum_n h_i(n) z^{-n}.$$

### A. Multiwavelet Filter Bank

As in the case of traditional wavelet bases, Mallat's algorithm associates a multiwavelet filter bank with a multiwavelet basis, as illustrated in Fig. 1. In Fig. 1, each component is an $r$-vector. The filter coefficients are $r \times r$ matrices, which, as in the traditional case, come from the dilation equation. For traditional wavelet bases, the discrete wavelet transform (DWT) is based on the corresponding filter bank structure. Similarly, the discrete multiwavelet transform (DMWT) will be based on the vector filter bank in Fig. 1. For this reason, the filter bank is of central importance.

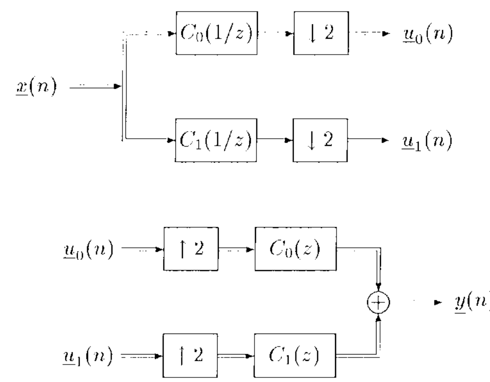


Fig. 1. Multiwavelet filter bank, analysis bank, and synthesis bank.

Although it has been suggested that multiwavelets and their filter banks might be well suited to multichannel data, in this paper, we are primarily interested in the design and application of multiwavelet systems for scalar discrete-time signals. For that reason, vector-valued sequences must be formed from scalar sequences. In this paper, this is done conceptually by grouping adjacent sets of $r$ samples. The notation for the vector valued sequence will be, for $r = 2$

$$\underline{x}(0) = \begin{pmatrix} x(0) \\ x(1) \end{pmatrix}, \quad \underline{x}(1) = \begin{pmatrix} x(2) \\ x(3) \end{pmatrix}, \text{etc.} \tag{7}$$

This is a critically sampled scheme. Notice that the expression $\underline{x}(n)$ denotes a vector sequence, whereas the expression $x(n)$ denotes a scalar sequence. This notation is used throughout the paper.

As noted by Rieder *et al.* [22], [23], the two-channel vector-valued filter bank in Fig. 1 can be redrawn as the four-channel *scalar*-valued filter bank in Fig. 2. The scalar sequence $u_0(n)$ in Fig. 1 and the vector sequence $\underline{u}_0(n)$ in Fig. 2 are related through (7). In general, the two-channel $r$-vector filter bank associated with a multiwavelet basis of multiplicity $r$ can be reorganized as a $2r$-channel scalar filter bank. The two-channel $r$-vector multiwavelet filter bank can also be described as a time-varying two-channel scalar filter bank [34]. Xia and Suter give a more thorough examination of vector filter banks and wavelets in [37].

This equivalence, which is obtained by reorganizing the filters, is very useful. It follows that the orthogonality conditions for the multiwavelet filter bank are the same as the orthogonality conditions for a $2r$-channel scalar filter bank. Here lies the similarity between multiplicity $r$ multiwavelet filter banks and $2r$-channel filter banks, the later being associated with $M$-band wavelet bases [14], [26]. The difference between these two systems lies in the way the filter bank is iterated on the lowpass branch. For four-band wavelets systems (one scaling function with three wavelet functions), the filter bank is applied again only to the first (lowpass) channel. In the multiwavelet case, the outputs of the first two channels are interlaced to form a single scalar sequence, and the filter bank is repeated on that single scalar sequence.

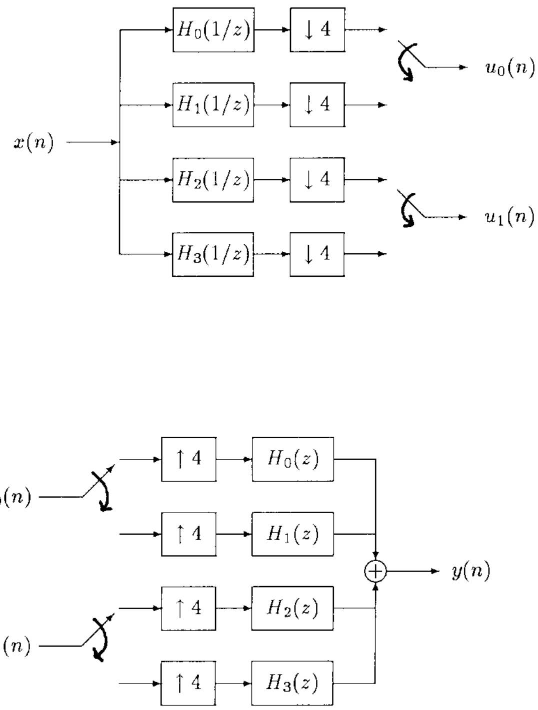


Fig. 2. Multiwavelet ($r = 2$) filter bank of Fig. 1 redrawn as a four-band scalar filter bank with interlacing/deinterlacing.

## III. Approximation Order and Prefiltering

One of the important properties of a wavelet basis is its approximation order. The approximation order of a traditional wavelet basis is given by the number of vanishing moments of $\psi(t)$. The same is true for a multiwavelet basis. If $\int t^k \psi_i(t)\, dt = 0$ for $i = 0, \cdots, r-1$ and $k = 0, \cdots, K-1$ (and not for $k = K$), then the multiwavelet basis is said to have approximation order $K$. For multiwavelets, conditions on the scaling coefficients for a specified approximation order has been developed in [2], [7], [12], [13], [19], [20], [22], and [28].

Consider a multiwavelet basis with approximation order $K$. If $f(t)$ is a polynomial of degree less than $K$, then $f(t)$ lies in the base scaling space $\mathcal{V}_0$. That means that a wavelet representation of $f(t)$ is sparse—it requires only scaling function coefficients. When $\phi(t)$ is of short support, then the same is (almost) true for piece-wise polynomial functions $f(t)$. Similarly, for functions that are *well modeled* as piecewise polynomials, the wavelet representation is efficient, in the sense that many coefficients in a wavelet expansion are close to zero. The ability of orthogonal wavelet bases to efficiently represent piecewise smooth functions is central to their success in estimation (denoising) and compression, and it depends on both approximation order and the short support of $\phi(t)$.

The construction of orthogonal multiwavelets having specified approximation order and short support has been studied by several authors [2], [4], [5], [7], [9], [22], [20], [21], [28], [30], [25]. Some of those examples are especially interesting because the functions are also symmetric, which is not possible for traditional (two-band) orthogonal real-valued wavelets of compact support (except for the Haar basis).

However, multiwavelet bases possessing approximation order $K$ lack some of the desirable properties traditional wavelet bases of the same approximation order posses. Consider an orthogonal wavelet basis based on a single scaling function. If the basis has approximation order $K$, then the basis has the following properties. The first property is essentially the definition.

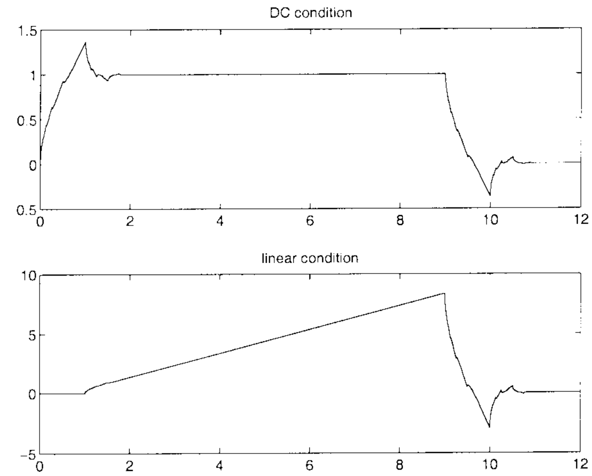


Fig. 3. Illustration of Property 2 with Daubechies' basis $D_2$.

**Wavelet bases based on a single scaling function: Approximation order $K$ properties**

1) **Zero moments—annihilation of $\mathcal{P}_k(R)$**

The moments of the wavelet vanish $\int t^k \psi(t)\, dt = 0$ for $k = 0, \cdots, K-1$.

2) **Preservation of $\mathcal{P}_k(R)$**

The discrete-time monomials $n^k$ $(k < K)$, when used as expansion coefficients with $\phi(t-n)$ gives a polynomial of degree $k$.

$$\sum_n n^k \, \phi(t-n) \in \mathcal{P}_k \tag{8}$$

For example, consider the Daubechies' wavelet basis $D_2$, with $K = 2$. Fig. 3 illustrates (8) for $k = 0$ and $k = 1$, where only a finite sum is shown (hence, edge behavior is nonpolynomial).

3) **Preservation of $\mathcal{P}_k(Z)$**

Discrete-time version of previous property. Let $h_0(n)$ and $h_1(n)$ be the scaling and wavelet filters. The output of the associated synthesis filter bank is

$$y(n) = ([\uparrow 2]u_0(n)) * h_0(n) + ([\uparrow 2]u_1(n)) * h_1(n)$$

where $u_0$ and $u_1$ are the two subband signals. If $u_0(n) = n^k, u_1(n) = 0$ and $k < K$, then $y(n)$ is a polynomial sequence of degree $k$. For example, Fig. 4 illustrates this property with $D_2$ for $k = 0$ and $k = 1$, where $u_0(n)$ is of finite support.

4) **Zero discrete moments—annihilation of $\mathcal{P}_k(Z)$**

The output of the highpass branch of the analysis filter bank is

$$u_1(n) = [\downarrow 2](h_1(n) * x(n)).$$

If the input is the monomial $x(n) = n^k$ with $k < K$, then the output of the highpass analysis channel is zero, and $u_1(n) = 0$. This property (the annihilation

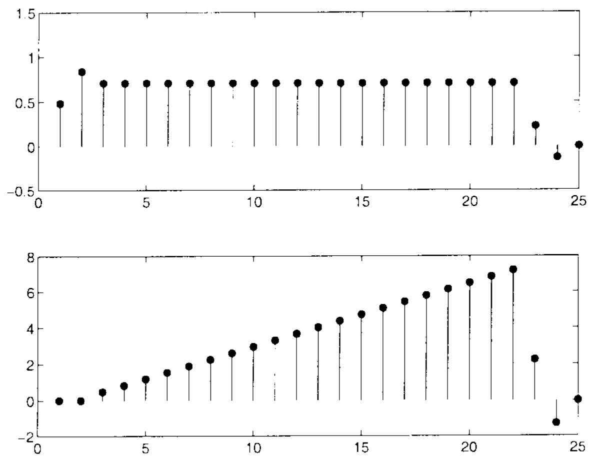

Fig. 4. Illustration of Property 3 with Daubechies' basis $D_2$.

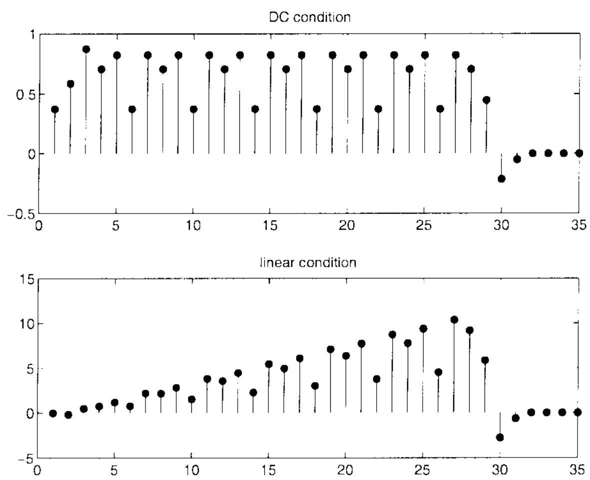


Fig. 5. For multiwavelets, Property 3 is not satisfied in general.

of discrete-time polynomials) is important because it is the reason efficient representations are obtained for piecewise smooth discrete-time functions.

For traditional wavelet bases, properties 2, 3, and 4 follow from approximation order $K$. Unfortunately, this is not true for multiwavelet bases in general. If a multiwavelet basis has approximation order $K$, then even though the moments of the wavelets vanish, the other properties are not satisfied.

For example, the DGHM multiwavelet basis [7], [9] has approximation order two; however, properties 2, 3, and 4 are not satisfied by the filter bank associated with that basis. Consider Property 3, and suppose $u_0(n) = n^k$ and $u_1(n) = 0$ in Fig. 2. Then, $y(n)$, which is shown in Fig. 5 for $k = 0$ and $k = 1$, is not a discrete-time polynomial of degree $k$. Consider also Property 2. Because the DGHM basis has approximation order two, linear polynomials can be represented in the base scaling space $\mathcal{V}_0$. However, the functions

$$\sum_n 1 \cdot \phi_0(t-n) + 1 \cdot \phi_1(t-n)$$
$$\sum_n (2n) \cdot \phi_0(t-n) + (2n+1) \cdot \phi_1(t-n)$$

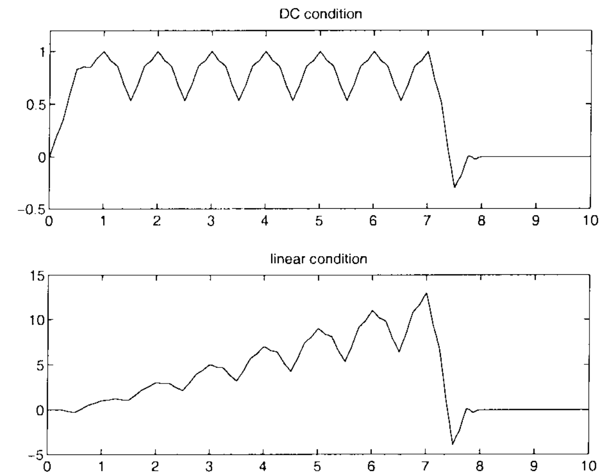


Fig. 6. For multiwavelets, Property 2 is not satisfied in general.

which are illustrated in Fig. 6, are not polynomials of degree 0 and 1, respectively. This demonstrates that for multiwavelet bases, the use of polynomial sequences as expansion coefficients with the scaling functions do not yield polynomials in general. To obtain a constant, the correct coefficients are $u_0(2n) = \sqrt{2}, u_0(2n+1) = 1$; to obtain a linear function, $u_0(2n) = \sqrt{2}(n + \frac{1}{2}), u_0(2n+1) = n + 1$ [2], [9].

Figs. 3–6 illustrate the differences in the meaning of approximation order for wavelet bases based on one and more than one scaling function. Although the DGHM scaling functions were employed here to show that for multiwavelets properties, 2, 3, and 4 do not follow from approximation order, other multiwavelet bases that have been described illustrate the same behavior.

Multiwavelet bases possessing properties 2, 3, and 4 are said to be order-$K$ balanced.

## A. Prefiltering

Because approximation order for multiwavelets is not accompanied by the additional properties discussed above, a preprocessing step is necessary to obtain an efficient signal representation, which is important for compression and denoising. Methods for preprocessing the discrete data, which is also called prefiltering or wavelet initialization, have been put forth for both traditional wavelet transforms [1], [27] and multiwavelet transforms [11], [24], [35], [36]. For example, Downie and Silverman have investigated the performance of several prefilters for multiwavelets with respect to denoising via thresholding and assess prefilters according to several different attributes (length, degree of approximation, and frequency response) [8].

For wavelet transforms based on a single scaling function, the prefiltering step is often omitted in many applications, without significant consequence. On the other hand, examples are given in [1] for which it is shown that prefiltering is needed. The importance of performing prefiltering for the traditional DWT evidently depends on the way in which the wavelet representation is used or interpreted. However, because of the difference noted above between approximation order for multiwavelet bases and traditional wavelet bases, a prefiltering step is fundamentally more essential when using a multiwavelet transform.

Although prefiltering alleviates some of the problems with multiwavelets, some reasons for wishing to avoid prefiltering are as follows.

1) If the prefiltering step does not constitute an orthogonal transform, then the orthogonality of the DMWT is "lost." The overall transformation of the original discrete-time data, including the preprocessing step, is no longer orthogonal. This may or may not be important depending on the application, but for many applications, orthogonality is desirable. In compression, quantization noise can be magnified by nonorthogonal postprocessing. In denoising, orthogonal transforms map white noise to white noise.
2) As noted above, many examples of orthogonal multiwavelet bases are most interesting for their symmetry properties. However, if the prefilter is not symmetric, then the symmetry properties of the DMWT are "lost" when prefiltering is included.
3) The use of prefiltering effectively increases the support of the basis functions if the prefilter has more than one nonzero coefficient. Single coefficient prefilters are described in [8], [31], and [36].

In the design of prefilters, it is desired that properties of the multiwavelet basis such as orthogonality, approximation order, short support, and symmetry be preserved as far as possible. For example, Hardin and Roach have constructed prefilters that preserve orthogonality and approximation order (up to $K = 3$) [11], [24].

However, in light of the issues noted above, it is natural to pursue the design of multiwavelet bases for which properties 2, 3, and 4 *are* satisfied (i.e., bases for which prefiltering can be avoided for most applications). Lebrun and Vetterli designed such multiwavelets for which they coined the term "balanced" [16], [17]. A multiwavelet basis with approximation order $K$ for which properties 2, 3, and 4 are satisfied is balanced in the sense that

$$\int (t - i/r)^k \, \phi_i(t)\, dt = \int \, (t - j/r)^k \, \phi_j(t)\, dt \qquad (9)$$

for $k = 0, \cdots, K-1; i, j = 0, \cdots, r-1$. In particular, for $k = 0$, we have $\int \phi_i(t)\, dt = \int \phi_j(t)\, dt$. For $k > 0$, the monomial $t^k$ must be shifted in (9) because $\int f(t)\, \phi_i(t)\, dt$ effectively samples $f(t)$ at $t = i/r + c + Z$.

A technique introduced in [16] and [17] applies a certain unitary transformation to the $r \times r$ scaling filter of an existing multiwavelet system. That method, as described in [16] and [17], achieves first-order balancing. The properties 2, 3, and 4 are satisfied for $k = 0$ but, unfortunately, not for greater $k$, even when the approximation order $K$ of the original multiwavelet system is greater than 1.

The following section characterizes orthogonal multiwavelets balanced up to their approximation order (for which properties 2, 3, and 4 are satisfied for $k$ up to $K - 1$). Orthogonal multiwavelet bases with approximation orders 2 and 3 are given, for which the filters $h_i(n)$ have differing supports.

## IV. Balancing Conditions

To obtain balanced orthogonal multiwavelet bases with compact support, the approach taken here designs an orthogonal filter bank with additional discrete-time approximation properties. By expressing the orthogonality and approximation conditions in terms of the filters $h_i(n)$, orthogonal multiwavelets can be obtained that possess the approximation properties 2, 3, and 4 (they are balanced up to the approximation order). Below, systems of nonlinear algebraic equations are solved using Gröbner bases to obtain orthogonal balanced bases having approximation orders 2 and 3.

A multiplicity $r$ multiwavelet basis, as noted above, is associated with a $2r$-channel scalar filter bank. The orthogonality conditions for a multichannel filter bank are well known, giving

$$\sum_n h_i(n)\, h_j(n + 2rl) = \delta(i-j)\, \delta(l) \qquad (10)$$

for $i, j = 0, \cdots, r-1$. Although these orthogonality equations are quadratic, the additional equations for approximation properties are linear.

To obtain approximation order $K$ and properties 2, 3, and 4, for an orthogonal multiwavelet basis, it is sufficient to impose, say, Property 3. Consider Property 3 for the simplest case $k = 0$. Referring to Fig. 2 $(r = 2)$, suppose $u_0(n) = 1$, and $u_1(n) = 0$; then, the output $y(n)$ is periodic with period 4. For this $u_0(n)$, the four values $y(n)$ takes are

$$\begin{aligned} &\sum_n \, h_0(4n) + h_1(4n) \\ &\sum_n \, h_0(4n+1) + h_1(4n+1) \qquad (11)\\ &\sum_n \, h_0(4n+2) + h_1(4n+2) \\ &\sum_n \, h_0(4n+3) + h_1(4n+3). \qquad (12)\end{aligned}$$

However, Property 3 requires that $y(n)$ be constant. Requiring the equality of these four sums is equivalent to requiring that the sum $H_0(z) + H_1(z)$ has zeros at $z = -1, i, -i$ (see [14] and [26]). That is, the condition for first-order balancing for $r = 2$ is

$$(z^{-3} + z^{-2} + z^{-1} + 1) \quad \text{divides} \quad H_0(z) + H_1(z). \qquad (13)$$

For general $r, y(n)$ takes on $2r$ values $\Sigma_n \, \Sigma_{i=0}^{r-1} \, h_i(2rn + k)$ for $k = 0, \cdots, 2r-1$. The equality of these $2r$ values gives the condition for first-order balancing

$$\left(\frac{z^{-2r} - 1}{z^{-1} - 1}\right) \quad \text{divides} \quad \sum_{i=0}^{r-1} H_i(z).$$

This property also arises in the case of $M$-band wavelet bases based on a single scaling function [14], [26]. In that case, the scaling filter must satisfy the same divisibility requirement.

The analysis of Property 3 for $k > 0$ is less simple; however, it is facilitated by examining the filter bank structure. For $k = 1$, it will be again shown that divisibility of a certain transfer function by a power of $(z^{-2r} - 1)/(z^{-1} - 1)$ is key (as in [14] and [26]).

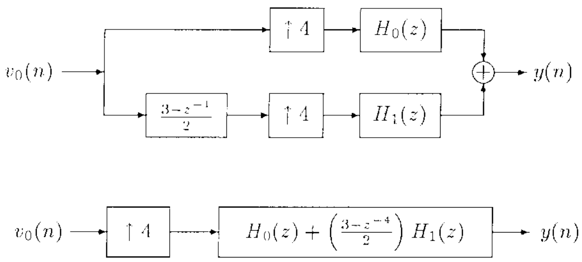


Fig. 7. Filters for second-order balancing condition.

Referring to Fig. 2, for $k = 1$ and $r = 2$, Property 3 requires that when $u_0(n) = n$ and $u_1(n) = 0$, then $y(n)$ must be a linear discrete-time polynomial (ramp function). Let $v_0(n) = u_0(2n)$ and $v_1(n) = u_0(2n+1)$. Note that $V_1(z)$ can be written (formally) as $V_1(z) = (3 - z^{-1}/2)V_0(z)$. For $k = 1, r = 2$, it follows that if Property 3 is to be satisfied, then the output of the first system shown in Fig. 7 must be a linear discrete-time polynomial. Using basic multirate operations [33], the first system in the figure can be redrawn as the second system shown. For the second system shown to map discrete-time linear polynomials (ramps) to themselves, the transfer function shown in the figure must have double roots on the unit circle at $z = -1, i, -i$. In other words, the condition for second-order balancing for $r = 2$ is

$$(z^{-3} + z^{-2} + z^{-1} + 1)^2 \quad \text{divides} \quad H_0(z) + \left(\frac{3 - z^{-4}}{2}\right) H_1(z). \tag{14}$$

Note that when $z = -1, i, -i$, the term $(3 - z^{-4}/2)$ equals 1. Therefore, if this condition for second-order balancing is satisfied, then it follows that (13) is also satisfied, i.e., second-order balancing implies first-order balancing, as we would hope. To obtain second-order balancing, it is necessary to ensure only (14) and not both (14) and (13).

For general $r$, the same procedure can be employed. We have $v_i(n) = u_0(rn+i)$ for $i = 0, \cdots, r-1$ and $V_i(z) = ((r + i/r) - (i/r)z^{-1})V_0(z)$. This gives the condition for second-order balancing as

$$\left(\frac{z^{-2r} - 1}{z^{-1} - 1}\right)^2 \quad \text{divides} \quad \sum_{i=0}^{r-1} \left(\frac{r+i}{r} - \frac{i}{r} z^{-2r}\right) H_i(z).$$

## V. Higher Order Balancing

For third-order balancing, the same approach used above can be employed again. This time, $u_0(n)$ in Fig. 2 is taken to be $n^2$, and $u_1(n) = 0$. Again, define $v_i(n) = u_0(rn+i) = (rn+i)^2$ for $i = 0, \cdots, r-1$. In this case, it was found that $V_i(z) = Q_i(z)V_0(z)$, where

$$Q_i(z) = \frac{1}{2r^2}\left((i+2r)(i+r) - 2i(i+2r)z^{-1} + i(i+r)z^{-2}\right).$$

The condition for third-order balancing is therefore

$$\left(\frac{z^{-2r} - 1}{z^{-1} - 1}\right)^3 \quad \text{divides} \quad \sum_{i=0}^{r-1} Q_i(z^{2r}) H_i(z). \tag{15}$$

TABLE I
CONDITIONS ON SCALING FILTERS $h_0(n)$ AND $h_1(n)$ FOR ORDER-$K$ BALANCING FOR $r = 2$

For order-1 balancing:

$(z^{-3} + z^{-2} + z^{-1} + 1)$ divides $H_0(z) + H_1(z)$.

For order-2 balancing:

$(z^{-3} + z^{-2} + z^{-1} + 1)^2$ divides $H_0(z) + \left(\frac{3-z^{-4}}{2}\right) H_1(z)$.

For order-3 balancing:

$(z^{-3} + z^{-2} + z^{-1} + 1)^3$ divides $H_0(z) + \left(\frac{15-10z^{-4}+3z^{-8}}{8}\right) H_1(z)$.

For order-4 balancing:

$(z^{-3} + z^{-2} + z^{-1} + 1)^4$ divides $H_0(z) + \left(\frac{35-35z^{-4}+21z^{-8}-5z^{-12}}{16}\right) H_1(z)$.

Following exactly the same procedure, we find the condition for order-$K$ balancing

$$\left(\frac{z^{-2r} - 1}{z^{-1} - 1}\right)^K \quad \text{divides} \quad \sum_{i=0}^{r-1} \frac{V_i(z^{2r})}{V_0(z^{2r})} H_i(z) \tag{16}$$

where $v_i(n) = (rn+i)^{K-1}$. With assistance from the computer algebra system *Maple*, we obtained a closed-form expression for $Q_i(z) = V_i(z)/V_0(z)$, which is described in the following characterization so obtained.

For order-$K$ balancing of multiplicity-$r$ orthogonal multiwavelets, we finally obtain the condition

$$\left(\frac{z^{-2r} - 1}{z^{-1} - 1}\right)^K \quad \text{divides} \quad \sum_{i=0}^{r-1} Q_i(z^{2r}) H_i(z) \tag{17}$$

where

$$Q_i(z) = \frac{P}{r^{(K-1)}(K-1)!} \sum_{l=0}^{K-1} \frac{(-1)^l}{(i+l\,r)} \binom{K-1}{l} z^{-l} \tag{18}$$

and

$$P = i \cdot (i+r) \cdot (i+2r) \cdots (i + (K-1)\cdot r). \tag{19}$$

An orthogonal multiplicity-$r$ multiwavelet system, with the scaling filters $h_i(n)$, for $i = 0, \cdots, r-1$, balanced up to its approximation order $K$ must satisfy this condition. Of course, it must also satisfy the usual orthogonality conditions required for the $2r$-channel filter bank in Fig. 2. In Table I, we enumerate the condition for the first few $K$ for the special case of $r = 2$ scaling functions. Note that in the special case $r = 1$, we retrieve the usual moment condition: $(z^{-1}+1)^K$ divides $H_0(z)$.

## VI. Examples

The preceding approach has been used to obtain a balanced orthogonal multiwavelet basis with approximation order 2. Certainly Daubechies' basis $D_2$ provides such a solution. In that case, $h_1(n) = h_0(n-2), \phi_0(t) = \phi_{D2}(2t)$, and $\phi_1(t) = \phi_{D2}(2t-1)$. It is interesting to see that Daubechies' basis is a special case. For that solution, the length of $h_0(n)$ and $h_1(n)$ are the both 4. However, there exists a balanced

TABLE II
BALANCED MULTIWAVELET BASIS ($r = 12$ WITH APPROXIMATION ORDER $K = 2$. $h_0$ AND $h_1$ ARE THE SCALING FILTERS, NORMALIZED SO THAT THEY SUM TO 1

| Scaling filters $h_0$ and $h_1$ |
|---|
| $A = \pm \frac{1}{2}\sqrt{-8 + 6\sqrt{3}}$ |
| $h_0(1) = -\frac{A^2}{6} + \frac{A}{6} + 1/4$ |
| $h_0(2) = -\frac{A}{3} + 1/3$ |
| $h_0(3) = \frac{A^2}{6} + \frac{A}{6} + \frac{5}{12}$ |
| $h_1(0) = -\frac{A^2}{24} - 1/48$ |
| $h_1(1) = \frac{A^2}{24} - \frac{A}{12} - 1/48$ |
| $h_1(2) = -\frac{A^2}{24} + \frac{A}{6} + 1/16$ |
| $h_1(3) = -\frac{A^2}{8} - \frac{A}{12} + \frac{7}{48}$ |
| $h_1(4) = \frac{A^2}{24} + \frac{25}{48}$ |
| $h_1(5) = \frac{A^2}{8} - \frac{A}{12} + \frac{13}{48}$ |
| $h_1(6) = \frac{A^2}{24} + \frac{A}{6} + \frac{5}{48}$ |
| $h_1(7) = -\frac{A^2}{24} - \frac{A}{12} - 1/16$ |

solution for which the length of $h_0(n)$ is less than 4 when the length of $h_1(n)$ is allowed to exceed 4. With $h_0(n)$ supported on $\{1, 2, 3\}$, the shortest length $h_1(n)$ for which a solution was obtained was of length 8 supported on $\{0, 1, 2, \cdots, 7\}$. For this length, two distinct real solutions were obtained (not counting negation of the scaling filters $h_0$ and $h_1$). The solutions were obtained by first forming the quadratic orthogonality constraints on $h_i(n)$ and the additional discrete-time approximation conditions of Property 3. The solutions to this system of nonlinear equations were then obtained using a lexical Gröbner basis [6] (for the computation of which the software *Singular* was employed [10]).

The two solutions are given in Table II, where the filters $h_0(n)$ and $h_1(n)$ have been normalized so that they sum to 1. The wavelet filters $h_2(n)$ and $h_3(n)$ are supported on $\{0, \cdots, 3\}$ and $\{0, \cdots, 7\}$, respectively. The scaling functions for which $A = -\frac{1}{2}\sqrt{-8 + 6\sqrt{3}}$ is illustrated in Fig. 8. For that solution, a pair of wavelets (which are not uniquely determined by $\phi_i(t)$ [29]) is also shown. The supports of $\phi_0(t)$ and $\phi_1(t)$ are $[0, 2]$ and $[0, 3]$. The supports of $h_i(n)$ do not indicate the exact supports of $\phi_i(t)$.

It is interesting to view the frequency responses of the filters $h_0(n)$ and $h_1(n)$, both of which are lowpass as shown in Fig. 9. Having only three coefficients, $h_0$ must be a crude lowpass filter. $h_1$ has a peculiar lowpass frequency response; however, notice that the frequency response of the sum $h_0 + h_1$ exhibits a more typical lowpass behavior. In addition, note the

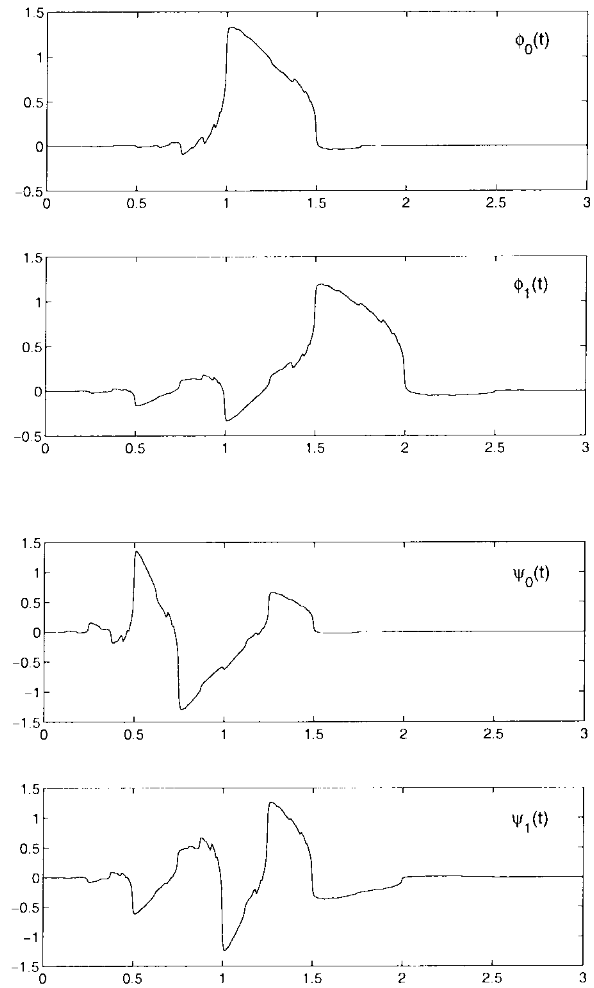


Fig. 8. Scaling and wavelet functions for which $A = -\frac{1}{2}\sqrt{-8 + 6\sqrt{3}}$ in Table II. The support of $\phi_0$ and $\psi_0$ is $[0, 2]$. The support of $\phi_1$ and $\psi_1$ is $[0, 3]$.

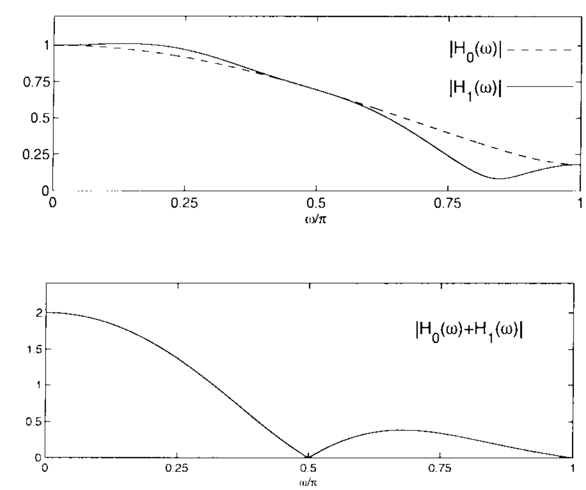


Fig. 9. Frequency response magnitude of filters $h_0, h_1, h_0 + h_1$, associated with Fig. 8.

zeros of the sum $h_0 + h_1$ at $\omega = \pi/2$ and $\omega = \pi$ in accordance with the preceding discussion. In Fig. 10, the zero plots of $H_0(z) + H_1(z)$ and $H_0(z) + (3 - z^{-4}/2)H_1(z)$ are shown, illustrating the divisibility conditions for first- and second-

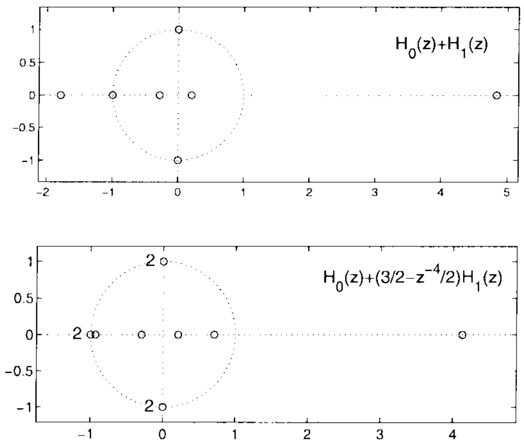


Fig. 10. Zero plots of $H_0(z) + H_1(z)$ and $H_0(z) + (3 - z^{-4}/2)H_1(z)$.

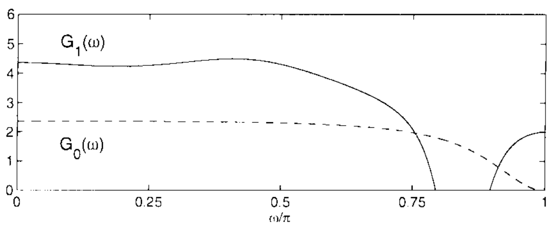


Fig. 11. Group delay of $h_0$ and $h_1$ associated with Fig. 8.

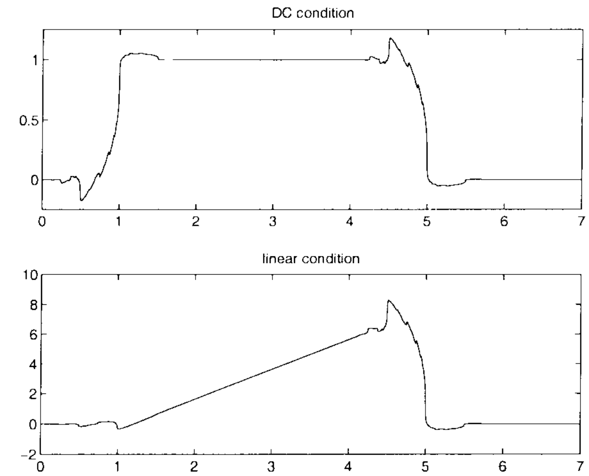


Fig. 12. Basis illustrated in Fig. 8 satisfies property 2. Illustration of dc and linear conditions.

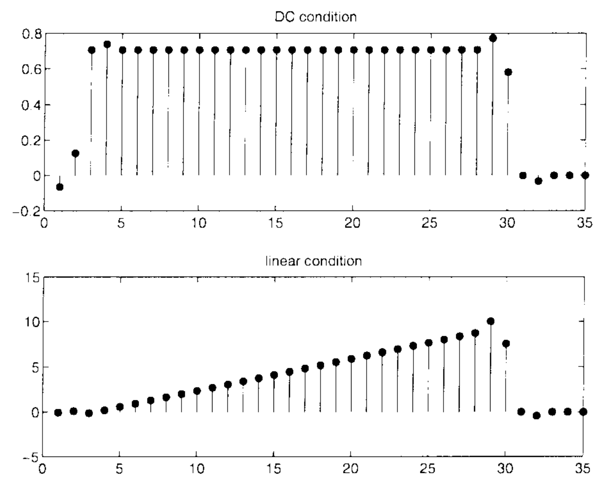


Fig. 13. Basis illustrated in Fig. 8 satisfies property 3. Illustration of discrete-time conditions.

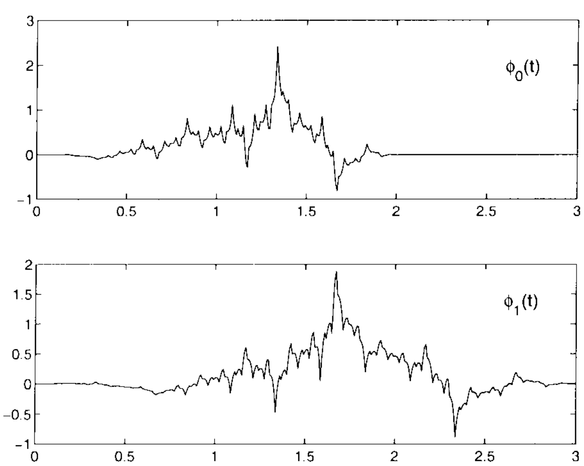


Fig. 14. Scaling functions for which $A = \frac{1}{2}\sqrt{-8 + 6\sqrt{3}}$ in Table II.

order balancing. Note the double zeros at $z = -1, i, -i$ in the second zero plot. (The zero at $z \approx -0.933$ is not to be confused with the double zero at $-1$; it is a separate zero.)

The group delay of the filters are shown in Fig. 11. It is interesting to note that the dc group delay of the two filters differs by 2. DC refers to the point $z = 1, \omega = 0$. This makes sense because $h_0$ and $h_1$ correspond to alternating samples. For wavelet bases based on a single scaling function, where $h_1(n) = h_0(n-2)$, the group delay differs by 2 for all frequencies $\omega$. The same is true for the DGHM basis. On the other hand, for general (unbalanced) multiwavelets, the difference of 2 at dc is not necessarily present.

As illustrated in Figs. 12 and 13, the approximation properties 2 and 3 are satisfied. Property 4 is also satisfied, although that is not illustrated here.

The scaling filter from Table II for which $A = \frac{1}{2}\sqrt{-8 + 6\sqrt{3}}$ is a second solution; however, the behavior of the associated scaling functions are quite different, as shown in Fig. 14. For this solution, although the frequency response of $h_0$ and $h_1$ are not shown here, they are hardly lowpass, explaining the nonsmooth characteristic of $\phi_0$ and $\phi_1$.

### *A. Multiplicity-4 Example*

Consider now a multiplicity-4 balanced orthogonal multiwavelet basis with approximation order 2. In this case, there are four scaling filters $h_0, \cdots, h_3$, and four wavelet filters $h_4, \cdots, h_7$.

Again, Daubechies' basis $D_2$ provides such a solution, with $h_1(n) = h_0(n-2), h_2(n) = h_0(n-4)$, and $h_3(n) = h_0(n-6)$. In that case, each is of length 4. The previous example provides another solution, with $h_2(n) = h_0(n-4)$ and $h_3(n) = h_1(n-4)$. In that case, the filters are of length 3, 8, 3, and 8, respectively. However, there exists a solution for which the support of $h_0(n)$ is $\{1, 2, 3\}$ and for which the maximum support of the scaling filters is *less* than 8.

TABLE III
BALANCED MULTIWAVELET SCALING FUNCTIONS ($r = 2$) WITH APPROXIMATION ORDER $K = 3$. $h_0$ AND $h_1$ ARE NORMALIZED SO THAT THEY SUM TO 1

$$h_0(0) = -\tfrac{B}{12} + \tfrac{A^2}{3} - \tfrac{5A}{4} + 1$$
$$h_0(1) = \tfrac{B}{4} - \tfrac{A^2}{2} + \tfrac{5A}{4}$$
$$h_0(2) = -\tfrac{B}{4} + \tfrac{A}{4}$$
$$h_0(3) = \tfrac{B}{12} + \tfrac{A^2}{6} - \tfrac{A}{4}$$
$$h_1(0) = \tfrac{B}{32} - \tfrac{7A^2}{64} + \tfrac{11A}{32} - 3/16$$
$$h_1(1) = -\tfrac{3B}{32} + \tfrac{13A^2}{64} - \tfrac{5A}{8} + \tfrac{21}{64}$$
$$h_1(2) = \tfrac{3B}{32} + \tfrac{A^2}{64} - \tfrac{9A}{32} + 1/2$$
$$h_1(3) = -\tfrac{B}{32} - \tfrac{3A^2}{64} - \tfrac{A}{8} + \tfrac{45}{64}$$
$$h_1(4) = \tfrac{B}{16} - \tfrac{9A^2}{32} + \tfrac{17A}{16} - 3/8$$
$$h_1(5) = -\tfrac{3B}{16} + \tfrac{11A^2}{32} - \tfrac{3A}{4} + \tfrac{7}{32}$$
$$h_1(6) = \tfrac{3B}{16} - \tfrac{A^2}{32} + \tfrac{A}{16}$$
$$h_1(7) = -\tfrac{B}{16} - \tfrac{5A^2}{32} + \tfrac{A}{2} - \tfrac{9}{32}$$
$$h_1(8) = -\tfrac{B}{96} + \tfrac{11A^2}{192} - \tfrac{5A}{32} + 1/16$$
$$h_1(9) = \tfrac{B}{32} - \tfrac{3A^2}{64} + \tfrac{A}{8} - \tfrac{3}{64}$$
$$h_1(10) = -\tfrac{B}{32} + \tfrac{A^2}{64} - \tfrac{A}{32}$$
$$h_1(11) = \tfrac{B}{96} + \tfrac{7A^2}{192} - \tfrac{A}{8} + \tfrac{5}{64}$$

$$A = 7/2 - \frac{\sqrt{17 \pm 2\sqrt{46}}}{2}$$

$$B = A^2 - \frac{12A}{5} + 3/5 \pm \frac{\sqrt{25A^4 - 710A^3 + 4019A^2 - 7462A + 3594}}{5}$$

Specifically, solutions were obtained for which the four scaling filters $h_0, \cdots, h_3$ have lengths 3, 6, 4, and 6, respectively.

As in the previous example, the scaling functions, although not shown here, exhibit differing degrees of smoothness. Interestingly, the scaling functions resemble Daubechies' scaling functions $\phi_{D2}$ more so than those shown in Fig. 8. This can be accounted for by noting the smaller disparity between the lengths of the scaling filters $h_i$ in this example, in comparison with the previous example. It is expected that for balanced orthogonal multiwavelets with higher multiplicity, designed with smaller length disparity, the multiple scaling functions will increasingly resemble Daubechies' scaling functions.

### B. Example with Approximation Order $K = 3$

In this example, we consider a multiwavelet basis with approximation order $K = 3$ with third-order balancing and multiplicity $r = 2$. By solving a multivariate polynomial system of equations with a lexical Gröbner basis, we obtained solutions for which $h_0(n)$ and $h_1(n)$ are of length 4 and 12, respectively, supported on $\{0, \cdots, 3\}$ and $\{0, \cdots, 11\}$. We found four real-valued solutions, which are given in Table III. For two of those solutions, the scaling functions are illustrated in Fig. 15. The other two solutions are not smooth. The solution illustrated by the top half of Fig. 15 closely resembles Daubechies' basis $D_3$. The other basis is smoother. Interestingly, the frequency responses of the scaling filters $h_0$ and $h_1$ for that basis have a "poorer" stopband behavior than the Daubechies-like basis (although the frequency responses are not shown here).

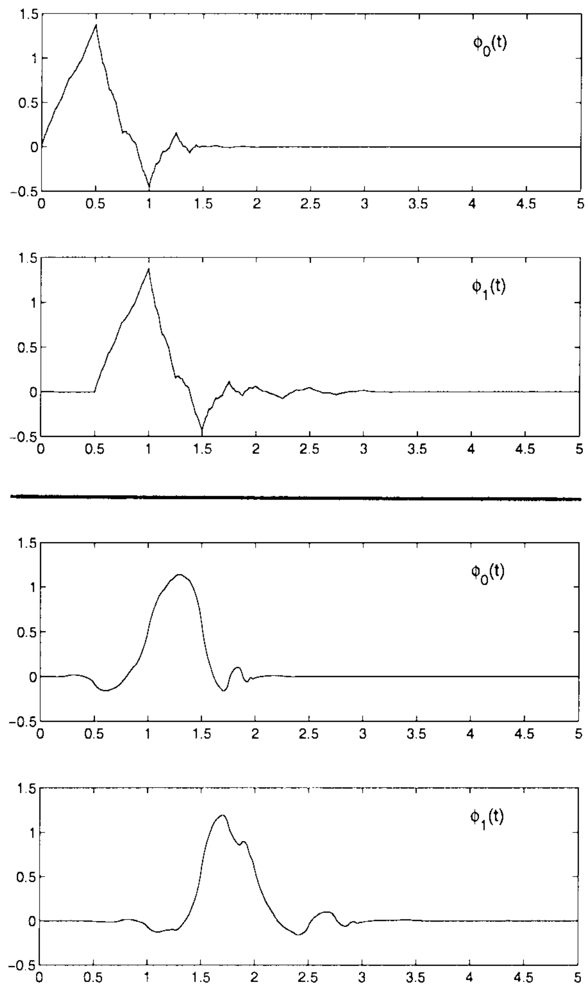


Fig. 15. Scaling functions for approximation order $K = 3$ example. For the top two scaling functions $A \approx 0.7357, B \approx -0.2235$ in Table III. For the lower two scaling functions $A \approx 2.5733, B \approx 0.6938$ in Table III. For both bases, the supports of $\phi_0(t)$ and $\phi_1(t)$ are $[0, 3]$ and $[0, 5]$, respectively.

## VII. SUPPORT/APPROXIMATION TRADE-OFF

Denoising via the nonlinear thresholding of wavelet coefficients is one of the successful applications of traditional wavelet bases and transforms. It is natural to consider the way in which the generalization provided by multiwavelets may be used to improve wavelet denoising. First, it is useful to note the following result regarding denoising with traditional wavelets. Wavelet denoising with a (shift-invariant) redundant wavelet transform can yield results that are superior to those obtained with a critically sampled orthogonal wavelet transform [3], [15]. It is expected that the same will be true for balanced multiwavelet systems. It should also be noted that the redundant transform was developed partly as a response to the lack of shift-invariance of wavelet filter banks and transforms. With such a transform, shift invariance is retrieved by effectively including all shifts of the data and comes at the expense of a redundant signal representation. In a sense, multiwavelet systems suffer from the loss of shift-invariance even more so than traditional wavelet systems (by a factor of $r$ for multiplicity $r$ multiwavelets). Finally, the use of a multiwavelet transform gives greater flexibility in managing the fundamental tradeoff between approximation order and time localization properties of scaling functions. For these reasons, extending the use of a redundant transform to multiwavelet bases to obtain a shift-invariant balanced multiwavelet transform (tight frame) may lead to effective denoising methods. Positive results for denoising using unbalanced multiwavelet transforms have been reported in [8], [31], and [32].

As noted above, both approximation order and short support are important properties of wavelet bases that make them successful for applications such as denoising and compression. The ideal scaling function would have short support (like the Haar basis) and high approximation order (like Daubechies' basis $D_8$). However, there is a fundamental tradeoff between support length and approximation order. Approximation order and short support are competing objectives in the design of wavelet bases. For wavelet bases based on a single scaling function, the length of the scaling filter is the single parameter with which we can manage the tradeoff between the support and the approximation order. However, by employing a wavelet basis based on more than one scaling function, it is possible to manage the support/approximation tradeoff with more flexibility by allowing the filters $h_i(n)$ to take on differing lengths. In particular, it is possible to construct a balanced multiwavelet basis for which the length of one of the filters, say $h_0(n)$, is less than the minimal length of a scaling filter of a traditional wavelet system of the same approximation order. This was achieved by extending the length of the remaining filter(s) $h_i(n)$, as expected, to maintain the approximation order.

Finally, we note that this paper did not consider the problem of constructing balanced multiwavelet bases with symmetric scaling functions; to see consideration of this problem, refer to [16] and [17].

## VIII. CONCLUSION

Using filter banks and discrete-time properties, this paper examined the differences between traditional wavelet bases and multiwavelet bases with equal approximation order. It was noted that multiwavelet bases lack properties traditional wavelet bases (of equal approximation order) possess. Those properties (preservation and annihilation of discrete-time polynomials) are important for processing discrete-time signals because without them, the transform based on the associated filter bank gives a representation that is not necessarily sparse. Although this problem can be partly overcome by using an appropriate prefilter, this paper investigated the properties and design of multiwavelets that do possess the properties that are normally absent. It was found that the characterization of multiwavelets that are "balanced" up to their approximation order is similar to the characterization of approximation order for $M$-band wavelets: Certain transfer functions must possess zeros at roots of unity. These balanced multiwavelets (so named in [16] and [17]) are expected to simplify the usage of multiwavelets in applications.

Recently, Lebrun and Vetterli have also addressed the characterization of orthogonal multiwavelets balanced up to their approximation order [18] and have found that the characterization given here is equivalent to a factorization of the matrix refinement mask $z$-transform of the matrix scaling filter $C(n)$.

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