SYMMETRIC NEARLY ORTHOGONAL AND ORTHOGONAL NEARLY SYMMETRIC WAVELETS

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الخلاصــة

سوف نُـناقـش في هذا البحث تصميم الموجات الصـغـيّرة المتعامدة شبه المتماثـلة ذات قـناتين من جهة والموجات الصغـيّرة المتماثـلة الشبه متعامدة من جهة أخرى. وسوف نستعمل في كلا نوعي الموجات الصـغـيّرة قواعد جروبنر كطريقة للتصميم، ونقدم أمثـلة على كلتا الحالتـيّن .

ABSTRACT

In this paper we discuss the designs of 2-channel orthogonal near symmetric wavelets on the one hand, and the symmetric near orthogonal wavelets on the other. In both types of wavelets we use the Gröbner bases design approach and present examples of both cases.

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1. INTRODUCTION

Wavelets based on 2-channel filterbanks have experienced wide use in the signal processing community. They have been used in such applications as noise removal and data compression. However, such filterbanks cannot be made both symmetric *and* orthogonal, except for the case of the Haar wavelet [1]. Symmetric wavelets can be designed once the requirement of orthogonality is dropped. Such wavelets have proved to be popular and have witnessed applications in image processing [2–6] as well as applications in the medical field, see for example [7–9]. In [10] Fujii and Hoefer explore the use of interpolating biorthogonal wavelets for solving time-dependent Maxwell's equations.

One desired property in image processing applications is energy conservation. Such a property is satisfied by orthogonal but not by biorthogonal filterbanks. However, it turns out to be possible to obtain symmetric filterbanks which are nearly orthogonal, as will be seen further in the paper. Another desirable property involves uncorrelated input $x(n)$ whereby we have $\sum_n x(n)x(k+n) = \delta(k)$. In the case of orthogonal filterbanks the resulting outputs remain uncorrelated, which in general is not the case with biorthogonal filterbanks [3]. By approximating orthogonality of symmetric filterbanks the input signal $x(n)$ remains nearly uncorrelated.

A large pool of papers has been dedicated to the theory and design of 2-channel biorthogonal wavelets. We mention only a few of the published papers in this section. In [11] and [4], biorthogonal filters with rational coefficients, and thus simpler to implement, are proposed. In [12], a class of 2-channel biorthogonal coiflets is designed with the wavelet and scaling function having different vanishing moments, and where a lowpass filter $H_0(\omega)$ is assumed to be given. The paper addresses the design of odd length as well as even length filters. The algorithm in [1] seeks to minimize the quantity $\int_{-\pi}^{\pi}[2 - |H_0(\omega)|^2 - |H_0(\omega+\pi)|^2]\, d\omega$ using one degree of freedom, where $H_0(\omega)$ is a lowpass biorthogonal filter. In [3], a biorthogonal pair of filters is suggested as symmetric filters approximating orthogonality. In [13] Saint-Martin *et al.* investigate the near-orthogonality of various symmetric filters.

Additionally, we address the issue of designing 2-band orthogonal near symmetric wavelets. The idea of 2-band orthogonal filters made almost symmetric has been investigated in [1, 14–16]. None of the coefficients of the proposed filters are strictly symmetric; rather, they are approximately so. In [1] I. Daubechies designs filters with vanishing moments not only for the wavelet $\psi(\cdot)$, but also for the scaling function $\phi(\cdot)$. The resulting filters possess a degree of asymmetry which decreases as the number of moments increases. In the more recent paper by Monzón and Beylkin [17], nearly interpolating properties in coiflet-like filters are sought in addition to approximate symmetry.

In this paper, we discuss symmetric biorthogonal wavelets with very nearly orthogonal behavior, where the lowpass filters h_0 and g_0 obey the properties

$$\sum_k h_0(k)h_0(k-2n) \approx \delta(n), \tag{1.1}$$

$$\sum_k g_0(k)g_0(k-2n) \approx \delta(n). \tag{1.2}$$

To this end, we design symmetric filters using Gröbner bases [18–20]. It will be shown that the resulting symmetric filters are significantly close to orthogonality. In addition, we consider the design of nearly symmetric orthogonal filters where we have a subset of exactly symmetric coefficients as an alternative to the published results. In this paper, by nearly symmetric we mean a filter h_0 with a subset of its coefficients being exactly symmetric, for example a filter of the form $h_0 = [a\ b\ b\ a\ c\ d]$ where the subset $[a\ b\ b\ a]$ is symmetric.

2. PROPERTIES AND CONDITIONS

We start by defining spaces $\mathcal{V}_j$ and $\widetilde{\mathcal{V}}_j$ having the following telescopic property

$$\cdots \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \cdots \subset \mathcal{V}_j \subset \mathcal{V}_{j+1} \subset \cdots,$$

$$\cdots \subset \widetilde{\mathcal{V}}_{-1} \subset \widetilde{\mathcal{V}}_0 \subset \cdots \subset \widetilde{\mathcal{V}}_j \subset \widetilde{\mathcal{V}}_{j+1} \subset \cdots.$$

Similarly, we define the spaces $\mathcal{W}_j$ and $\widetilde{\mathcal{W}}_j$ as follows:

$$\mathcal{V}_{j+1} = \mathcal{V}_j \cup \mathcal{W}_j,$$

$$\widetilde{\mathcal{V}}_{j+1} = \widetilde{\mathcal{V}}_j \cup \widetilde{\mathcal{W}}_j.$$

We now can define the above mentioned spaces as follows:

$$\mathcal{V}_j = \operatorname*{Span}_k\{\phi(2^j t - k)\}, \qquad \widetilde{\mathcal{V}}_j = \operatorname*{Span}_k\{\widetilde{\phi}(2^j t - k)\},$$

$$\mathcal{W}_j = \operatorname*{Span}_k\{\psi(2^j t - k)\}, \qquad \widetilde{\mathcal{W}}_j = \operatorname*{Span}_k\{\widetilde{\psi}(2^j t - k)\}.$$

From the nesting property of the spaces discussed above we obtain

$$\phi(t) = \sqrt{2}\sum_n h_0(n)\phi(2t-n), \qquad \widetilde{\phi}(t) = \sqrt{2}\sum_n g_0(n)\widetilde{\phi}(2t-n),$$

$$\psi(t) = \sqrt{2}\sum_n h_1(n)\psi(2t-n), \qquad \widetilde{\psi}(t) = \sqrt{2}\sum_n g_1(n)\widetilde{\psi}(2t-n).$$

In addition, we impose biorthogonality condition whereby the spaces $\mathcal{V}_j$ and $\widetilde{\mathcal{W}}_j$ are orthogonal, as are the spaces $\widetilde{\mathcal{V}}_j$ and $\mathcal{W}_j$. Or we have

$$\mathcal{V}_j \perp \widetilde{\mathcal{W}}_j, \tag{2.1}$$

$$\widetilde{\mathcal{V}}_j \perp \mathcal{W}_j. \tag{2.2}$$

It can be shown that in order to obtain lowpass filters h_0 and g_0 satisfying perfect reconstruction condition the following equations need to be satisfied [21]:

$$G_0(z)H_0(z) + G_1(z)H_1(z) = 2z^{-m}, \tag{2.3}$$

$$G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0. \tag{2.4}$$

where the term z^{-m} accounts for the filters' causality. One can show that Equations (2.3) and (2.4) lead to $g_1(n) = (-1)^n h_0(N_h - n - 1)$ and $h_1(n) = (-1)^n g_0(N_g - n - 1)$, where $N_h = \text{length}\, h_0$ and $N_g = \text{length}\, g_0$, or we obtain the following biorthogonality condition:

$$\sum_k g_0(k)\ h_0(2n-k) = \delta(n).$$

Orthogonality then becomes a special case of the filters described in Equations (2.3) and (2.4), resulting in the orthogonality condition

$$\sum_k h_0(k)h_0(k-2n) = \delta(n).$$

The corresponding highpass filter h_1 is then given by

$$h_1(n) = (-1)^n h_0(N-1-n)$$

where $N = \text{length}\, h_0$.

2.1. Coifman Properties

A lowpass filter h_0 satisfies Coifman properties if its z-transform evaluated on the unit circle $H_0(\omega) = \sum_n h_0(n)e^{-j\omega n}$ possesses the following property:

$$H_0^{(p)}(0) = \delta(p),\ p : 0 \dots P \tag{2.5}$$

or equivalently

$$\int_{-\infty}^{\infty} t^p \phi_h(t)\,\mathrm{d}t = \delta(p),\ p : 0 \dots P.$$

In this paper we assume that both biorthogonal filters $G_0(z)$ and $H_0(z)$ have the same number P of zero derivatives at $\omega = 0$, as well as the same number of wavelet vanishing moments K.

3. FILTERS LENGTHS

It will be shown that the lengths of the various filters discussed in this paper depend directly on the properties being imposed. We consider separately the lengths of the symmetric filters resulting from two distinct approaches, as well as the lengths of the orthogonal filters approximating symmetry.

3.1. Symmetric Coiflets

Given lowpass filters h_0 and g_0 satisfying the biorthogonality conditions, we look for the filter supports N_h and N_g as functions of K and L, where we have a total of $N_h + N_g$ degrees of freedom. Thus, in addition to K zeros at $z = 1$ for each of h_0 and g_0, we require that the first L derivatives of $H_0(z)$ and $G_0(z)$ be zero at $z = 1$. Due to symmetry, the latter condition is automatically satisfied for odd derivatives. This can be seen from the general form of the odd length symmetric filter $H_0(\omega) = \sum_n b_n \cos(\omega n)$ ($H_0(\omega) = \sum_n c_n \cos(\omega(n-\frac{1}{2}))$ for even length case). Clearly, the odd derivatives of $H_0(\omega)$ with respect to ω are given only in terms of $\sin(\omega n)$ ($\sin(\omega(n-\frac{1}{2}))$ for the case of even length), resulting in $H_0^{(k)}(0) = 0,\ k\,\text{odd}$. In addition, imposing L zero derivatives at $z = 1$ ($\omega = 0$) on $H_0(z)$ results in L zero derivatives at $z = 1$ for $G_0(z)$ as well. Therefore we need just $L/2$ equations to guarantee the zero derivative condition. In addition, the biorthogonality conditions require $(N_h + N_g)/4$ equations. And, finally, symmetry requires $(N_h + N_g - 2K)/2$. Or, we obtain

$$2K + L/2 + (N_h + N_g)/4 + (N_h + N_g - 2K)/2 = N_h + N_g,$$

resulting in

$$4K + 2L = N_h + N_g. \tag{3.1}$$

Now, biorthogonality requires that both filters be either of even length, or of odd length, or we have

$$N_h - N_g = 2l, \; l \in \mathbb{Z}$$

and substituting $N_h = N_g + 2l$ in Equation (3.1) we obtain

$$N_g = 2K + L - l,$$

$$N_h = 2K + L + l.$$

For the case K is odd, the filters' lengths are necessarily odd, due to symmetry requirement. Then in this case we have l odd. Similarly, when K is even l takes on even values.

3.2. **Equal Coefficients Subsets**

As an alternative to Coifman filters, we consider the case where h_0 and g_0 have subsets of exactly equal coefficients, thus approximating orthogonality while maintaining symmetry. In other words, we look for biorthogonal filters h_0 and g_0 such that

$$h_0(n) = g_0(n), \qquad n : 0 < l \leq n \leq N - l. \tag{3.2}$$

For example, one possibility is h_0 and g_0 being of even lengths with a coefficient arrangement as follows:

$$h_0 = [h_0(0)\; h_0(1) \quad \mathrm{a\;\; b\;\; b\;\; a} \quad h_0(1)\; h_0(0)]\,,$$

$$g_0 = [g_0(0)\; g_0(1) \quad \mathrm{a\;\; b\;\; b\;\; a} \quad g_0(1)\; g_0(0)]\,, \tag{3.3}$$

with $l = 2$, and clearly the filters h_0 and g_0 share four coefficients. Now, if we consider the case L even, then the problem is similar to the preceding Coifman case, except that now instead of L zero derivatives at $z = 1$, we look for L coefficients shared by both h_0 and g_0. For L even, this results in $L/2$ equations. Or we now have

$$2K + L/2 + (N_h + N_g)/4 + (N_h + N_g - 2K)/2 = N_h + N_g$$

and

$$4K + 2L = N_h + N_g,$$

or we have

$$N_g = 2K + L - l,$$

$$N_h = 2K + L + l.$$

The case L odd results in similar equations, with

$$N_g = 2K + L - l + 1,$$

$$N_h = 2K + L + l - 1.$$

Notice that in this case even value of K implies odd value of L and thus l itself must take on odd values. Likewise, odd value of K entails even value of L, with l taking on even values so as to maintain an overall even length.

3.3. **Orthogonal Near Symmetric**

For a near symmetric lowpass filter with K zeros at $z = -1$ and an overall length of N coefficients and L symmetric coefficients we consider the minimum length of a filter satisfying orthogonality as well as near symmetry. We need $\frac{N}{2}$ coefficients to satisfy orthogonality condition for the 2-channel case and K coefficients for regularity condition [22]. This leaves us with $N - N/2 - K = N/2 - K$ degrees of freedom for symmetric coefficients.

For $L \in 2\mathbb{N}$ we have

$$N - N/2 - L/2 - K = (N - L)/2 - K,$$

and we have

$$N = 2K + L, \qquad L \in 2\mathbb{N}. \tag{3.4}$$

Similarly, for $L \in 2\mathbb{N} + 1$ we have

$$N = 2K + L - 1, \qquad L \in 2\mathbb{N} + 1. \tag{3.5}$$

Therefore, the support of $h_0(n)$ depends directly on the regularity K and the number of symmetric coefficients in h_0. However, one exception to the above equations stands out for the case of $K = 2$, length $h_0 = 8$, and $L = 6$. In theory, the filter can be made as symmetric as one wishes but always with $2K$ non-symmetric coefficients when L is even, and $2K - 1$ non-symmetric coefficients when L is odd.

We have found that, in general, for $L \leq K$ the symmetric coefficients have little effect on the overall shape of the filter frequency response, as will be shown below. It is only when L exceeds K that one begins to see some symmetry. This partly explains the rather large population of near-symmetric filters for $K = 2$ and the more limited one for $K = 4$.

3.4. **Scaling Functions Smoothness**

One desirable property of scaling functions is a high degree of smoothness ν_2 for a given K. It is shown in [23] that the highest possible derivative for a scaling function $\phi(\cdot)$, given the corresponding $h_0(n)$, is bounded by $\nu_2 < K$. Smoothness is measured using the Sobolev exponent of a scaling function ϕ defined as [24, 25]:

$$\nu_2(\phi) := \sup\{\nu_2 : \int_{-\infty}^{\infty} |\Phi(\omega)|^2 (1 + |\omega|^2)^{\nu_2} \mathrm{d}\omega < \infty\}.$$

The actual computation of ν_2 is found using [26], and for the normalization $\sum_n h_0(n) = \sqrt{2}$ we have

$$\nu_2 = -\frac{1}{2} \log_2 \lambda_{\max}$$

where $\lambda_{\max}$ is the largest eigenvalue of a matrix generated by $(c_{2i-j})_{-N \leq i,j \leq N}$ with $c(z) = Q_0(z)Q_0(z^{-1})$ and $Q_0(z)$ is known from $H_0(z) = (1 + z^{-1})^{K_0} Q_0(z)$.

3.5. **Near Orthogonality Criterion**

We will be addressing how closely the designed symmetric filters are to being exactly orthogonal. It will be seen that there is an extent of near orthogonality to the filters' even shifts with respect to themselves as well as to other filters. This in turn reflects the degree of orthogonality between the scaling function ϕ and the wavelets

ψ_i and their integer shifts. We will use $\theta(h_i, h_j, m)$ to indicate the angle between two vectors h_i and h_j shifted with respect to each other by m, defined as

$$\theta(h_i, h_j, m) = \arccos\left(\frac{< h_i(n), h_j(n-m) >}{\|h_i\| \cdot \|h_j\|}\right).$$

The results are to be tabulated for various filters h_i and their relative shifts.

3.6. Measure of Symmetry

Exploiting the fact that symmetric filters are associated with linear phase, we look at the group delay as a measure of symmetry approximation for the case of orthogonal filterbanks. A filter $H_0(\omega)$ group delay is given by

$$\tau(\omega) = -\frac{\mathrm{d}\theta(\omega)}{\mathrm{d}\omega}$$

where $\theta(\omega)$ is the phase of $H_0(\omega)$. Various definitions of phase distortion have been used in [1, 16, 27]. Define the group delay error e as follows:

$$e = \int_0^{\pi/2} |\tau(\omega) - \tau_0| \, \mathrm{d}\omega \tag{3.6}$$

where τ_0 is the mean group delay over the interval $[0, \frac{\pi}{2}]$. The integral is evaluated only over the passband as the group delay behavior over the stopband is of little relevance. Equation (3.6) can be approximated as a summation,

$$e \simeq \frac{1}{2N} \sum_{n=0}^{N-1} \left|\tau\left(\frac{\pi n}{2N}\right) - \tau_0\right| \tag{3.7}$$

where we have N points equally distributed over $\left[0, \frac{\pi}{2}\right]$ and τ_0 is the mean value defined as $\frac{1}{2N}\sum_{n=0}^{N-1} \tau(\frac{\pi n}{2N})$. Obviously, given a dyadic nontrivial orthogonal filter h_0, the more symmetric coefficients we have, the closer e is to zero, with $e = 0$ achieved only when the filter h_0 is exactly symmetric. This of course cannot be achieved with the 2-band orthogonal filterbanks. Thus, for such filterbanks we seek h_0 with $e \approx 0$.

4. EXAMPLES

In this section we discuss filters generating wavelets and scaling functions of various properties. We discuss two examples illustrating the two design approaches resulting in symmetric and approximately orthogonal filters, as well as an example of an orthogonal filterbank approximating symmetry.

4.1. Example 1

Consider the case of symmetric near orthogonal filter design where we impose the following constraints: both filters h_0 and g_0 have $K = 5$, and both filters have the first eight derivatives equal to zero at $\omega = 0$, or we need to satisfy the following conditions:

- $(1 + z^{-1})^5 \mid H_0(z)$,
- $(1 + z^{-1})^5 \mid G_0(z)$,
- $G^{(k)}(\omega)|_{\omega=0} = H^{(k)}(\omega)|_{\omega=0} = 0, \ 1 \le k \le 8$.

We thus obtain filters of identical lengths with the coefficients as given in Table 1. Notice in Figure 1 the high degree of similarity of the resulting filters and the associated scaling functions. Additionally, the filters span spaces which are largely orthogonal, as can be seen in Table 2 where the angles are close to 90 degrees, and where the angle between h_0 and g_0 is small, reflecting the similarity of the filters. The resulting scaling functions $\{\phi_h, \phi_g\}$ are differentiable at least twice, with $\nu_2(\phi_h) \approx 2.3472$ and $\nu_2(\phi_g) \approx 2.0338$.

Table 1. The Coefficients for Example 4.1, with $K = 5$, $L = 8$.

n	$h_0(n)$	$g_0(n)$
0,17	0.0001605988	0.0002809102
1,16	0.0002633873	–0.0004607019
2,15	–0.0028105671	–0.0014760379
3,14	–0.0022669755	–0.0016765216
4,13	0.0246782363	0.0192309116
5,12	–0.0061453735	–0.0001723898
6,11	–0.1137025792	–0.1099707039
7,10	0.1226794070	0.1091804942
8,9	0.6842506470	0.6921708203

Table 2. Angles Between Spaces Generated by Filters in Table 1, with $K = 5$, $L = 8$.

shift	h_0, h_0	h_1, h_1	h_0, h_1	h_0, g_0	h_0, g_1
0	0	0	90	1.41	90
2	89.70	89.69	89.22	90	90
4	89.85	89.86	89.46	90	90
6	89.96	89.98	89.94	90	90
8	89.99	89.98	89.96	90	90
10	89.99	89.99	89.99	90	90
12	89.99	89.99	89.99	90	90
14	89.99	90	89.99	90	90

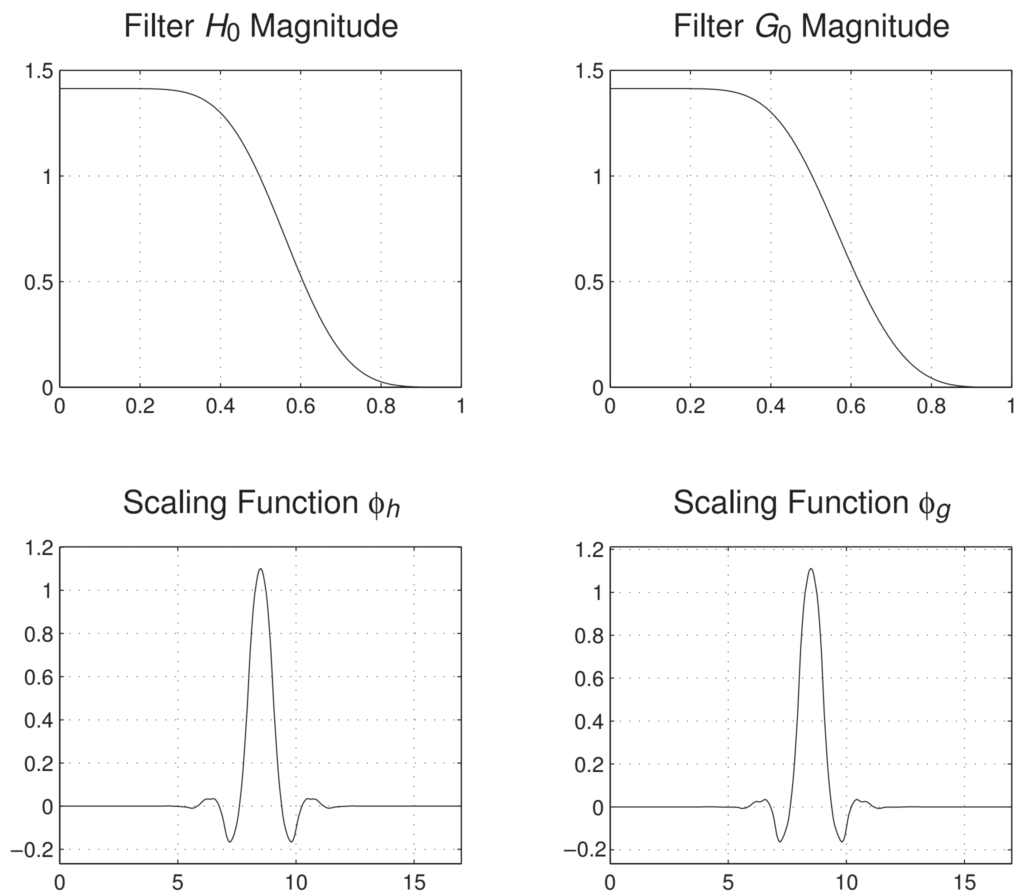

Figure 1. Case $K = 5,\ L = 8$, with filter coefficients listed in Table 1.

4.2. **Example 2**

In this example we address the design of $K = 3$ symmetric nearly orthogonal filterbanks by requiring that the filters h_0 and g_0 share six coefficients, or we impose the following conditions:

- $(1 + z^{-1})^3 \mid H_0(z)$,
- $(1 + z^{-1})^3 \mid G_0(z)$,
- $h_0(n) = g_0(n),\ 3 \leq n \leq 8$.

The resulting filters are of length $N = 12$ for both h_0 and g_0. Two distinct sets of filters result, with one solution offering near orthogonality, and the resulting coefficients tabulated in Table 3. The filters, depicted in Figure 2, along with the resulting scaling functions, are highly orthogonal, as suggested in Table 4. From the table, it is clear that h_0 and g_0 are separated by a very small angle, namely 0.5853, suggesting how similar the filters are. The angles made by the filters and their even shifts are tabulated in Table 4. Clearly, the angles are very close to 90 degrees, indicating how close to orthogonality the filters are. The smoothness coefficients are given by $\nu_2(\phi_h) \approx 1.4843$ and $\nu_2(\phi_g) \approx 1.4412$.

4.3. **Example 3**

An orthogonal 2-channel filterbank with a lowpass filter with $K = 2$ and the first six coefficients to be exactly symmetric is sought, or we design an orthogonal lowpass filter h_0 with the following properties:

- $(1 + z^{-1})^2 \mid H_0(z)$,
- $h_0(n) = h_0(5 - n),\ 0 \le n \le 5.$

The relevant equations describing the filter's behavior are given in Table 5, where we list the equations following the tradition of dropping the right hand side, which is implicitly understood to be zero. It is clear from the table

Table 3. Symmetric Lowpass Filters with $K = 3$.

n	$h_0(n)$	$g_0(n)$
0	−0.0019128844	0.0025454063
1	0.0033707110	0.0044852837
2	0.0092762126	0.0037033492
3	**−0.0855138167**	**−0.0855138167**
4	**0.0851905285**	**0.0851905285**
5	**0.6966960301**	**0.6966960301**
6	**0.6966960301**	**0.6966960301**
7	**0.0851905285**	**0.0851905285**
8	**−0.0855138167**	**−0.0855138167**
9	0.0092762126	0.0037033492
10	0.0033707110	0.0044852837
11	−0.0019128844	0.0025454063

Table 4. Angles Between Spaces Generated by Filters h_0 and g_0.

shift	$h_0,\ h_0$	$g_0,\ g_0$	$h_0,\ h_1$	$h_0,\ g_0$	$h_0,\ g_1$
0	0	0	90	0.58	90
2	89.97	89.96	89.98	90	90
4	89.84	89.84	89.75	90	90
6	89.78	89.78	89.80	90	90
8	89.97	89.97	89.97	90	90
10	89.99	89.99	89.99	90	90

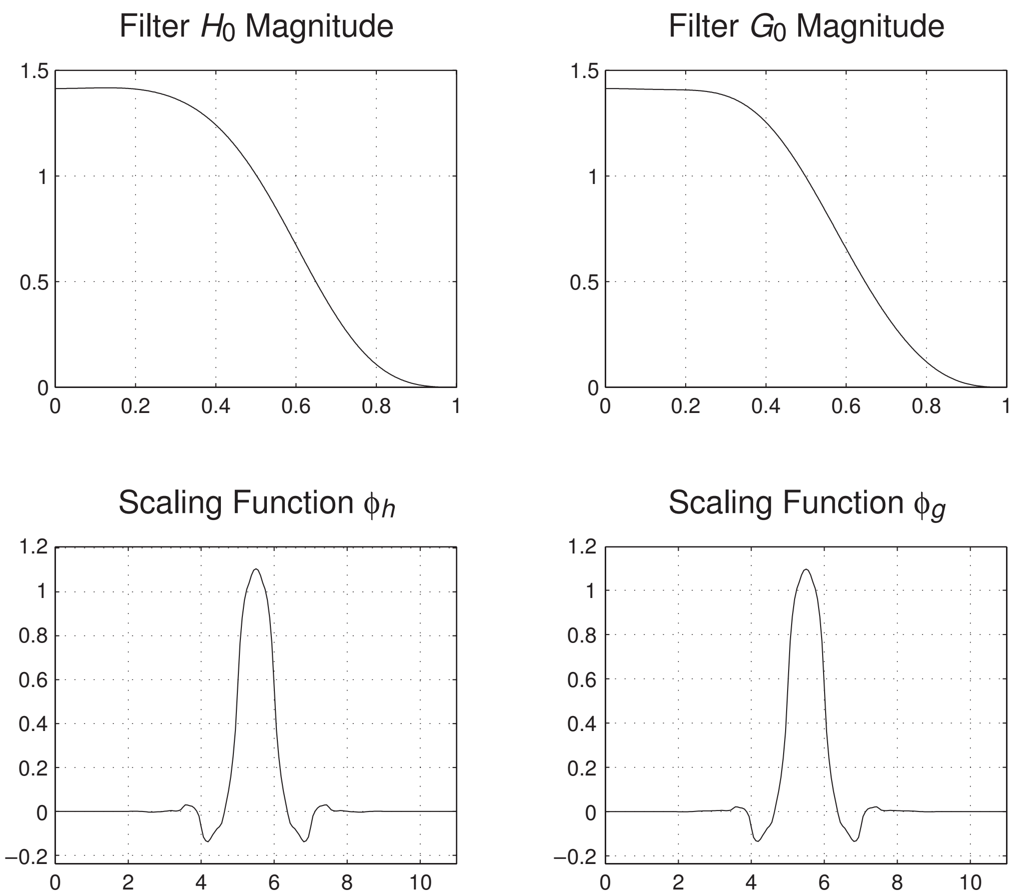

Figure 2. $K = 3$, length $h_0 =$ length $g_0 = 12$, $\nu_2(\phi_h) \approx 1.4843$, $\nu_2(\phi_g) \approx 1.4412$.

that the nonlinearity of the equations arises from the orthogonality condition. The resulting Gröbner basis is listed in Table 6. Clearly, we have one nonlinear equation in one unknown A, and the remaining equations being linear in the unknown. Then it is straightforward to solve the resulting system of equations. To this end two filters have resulted with six symmetric coefficients. The filter in question has the following coefficients, with $A = -\frac{\sqrt{2}}{4} \pm \frac{\sqrt{30}}{16}$:

$$h_0 = \left[\underbrace{-\frac{\sqrt{2}}{16} \quad \frac{\sqrt{2}}{16} \quad A + \frac{\sqrt{2}}{2} \quad A + \frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{16} \quad -\frac{\sqrt{2}}{16}}_{\text{Symmetric coefficients}} \quad -A \quad -A \right]. \tag{4.1}$$

We consider the case $A = -\frac{\sqrt{2}}{4} + \frac{\sqrt{30}}{16}$. To this end, Figure 3 indicates an extent of symmetry of the resulting scaling function. The group delay of the corresponding filter h_0 reveals a degree of flatness, showing an extent of symmetry. The scaling function possesses a smoothness coefficient of $\nu_2(\phi_h) \approx 1.5094$. The zero located at $z \approx -0.9004$ appears to contribute to the scaling function's smoothness. As for the symmetry error parameter, e, we obtain $e \approx 0.0191$. It is possible to reduce the error by increasing the number of symmetric coefficients of h_0. Indeed, by making more symmetric coefficients available, the error e can be made arbitrarily close to zero.

Table 5. Equations Describing an Orthogonal Nearly Symmetric Lowpass Filter with $K = 2$ and Length $h_0 = 8$.

Partial Symmetry
$h_0(0) - h_0(5)$
$h_0(1) - h_0(4)$
$h_0(2) - h_0(3)$
2-Regularity
$h_0(0) - h_0(2) - 3h_0(4) - 5h_0(6) + 6h_0(7) + 4h_0(5) + 2h_0(3)$
$h_0(1) + 3h_0(3) + 5h_0(5) + 7h_0(7) - 6h_0(6) - 4h_0(4) - 2h_0(2)$
Orthogonality
$h_0(1)h_0(3) + h_0(5)h_0(7) + h_0(3)h_0(5) + h_0(0)h_0(2) + h_0(2)h_0(4) + h_0(4)h_0(6)$
$h_0(3)h_0(7) + h_0(1)h_0(5) + h_0(0)h_0(4) + h_0(2)h_0(6)$
$h_0(0)h_0(6) + h_0(1)h_0(7)$
Normalization
$M^2 - 2$
$h_0(0) + h_0(1) + h_0(2) + h_0(3) + h_0(4) + h_0(5) + h_0(6) + h_0(7) - M$

Table 6. Gröbner Basis of Lowpass Filter with $K = 2$, $M = \sqrt{2}$, and $A = h_0(7)$.

$128A^2 - 64AM + 1$
$h_0(6) - A$
$16h_0(5) + M$
$16h_0(4) - M$
$2h_0(3) + 2A - M$
$2h_0(2) + 2A - M$
$16h_0(1) - M$
$16h_0(0) + M$

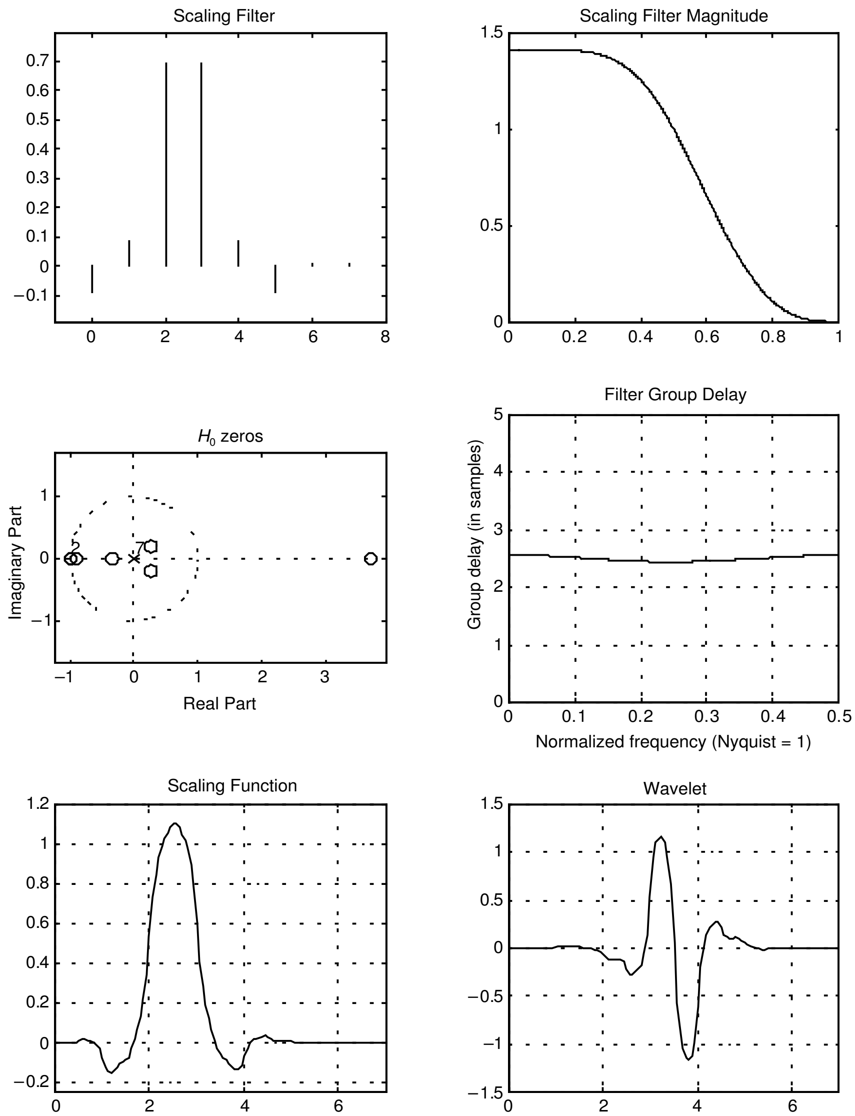

Figure 3. $K = 2$, $L = 6$, $e \approx 0.0191$, and $\nu_2(\phi_h) \approx 1.5094$

5. CONCLUSION

Using Gröbner bases method, it was possible to design two types of 2-channel wavelets. In one case we have designed symmetric filters approximating orthogonality, and in the other we have designed orthogonal filters approximating symmetry. The resulting symmetric filters can be made to approximate orthogonality arbitrarily, while the orthogonal filters can be as close to symmetry as is wished. Indeed, the flexibility offered by the design method allows the construction of filterbanks with various properties with relative ease.

ACKNOWLEDGMENTS

The authors would like to express their sincere thanks to the paper reviewers. This material is based upon work supported by the National Science Foundation under Grant No. 9875452.

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Paper Received 1 October 2002; Revised 31 May 2003; Accepted 11 June 2003.