Maximally Flat Lowpass FIR Filters with Reduced Delay*

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January 10, 1997

Abstract

This paper describes a new class of nonsymmetric maximally flat lowpass FIR filters. By subjecting the magnitude and group delay responses (individually) to differing numbers of flatness constraints, the new filters are obtained. It is found that these filters achieve a smaller delay than symmetric filters while maintaining relatively constant group delay around $\omega = 0$, with no degradation of the frequency response magnitude. The design of these filters is initially investigated using Gröbner bases. An analytic design technique, applicable to a subset of the forgoing filters, is provided that does not depend on Gröbner basis computations.

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`Submitted to: IEEE Transactions on Circuits and Systems II - March 1996`
`Accepted December 1996`

*This work has been supported by BNR and by NSF grant MIP-9316588.

List of Figures

List of Tables

1 Introduction

A fair amount of attention has been focused on the design of nonsymmetric digital FIR filters. Laakso et al. discuss in [12] the design of nonsymmetric FIR filters for fractional delay applications. Lagrangian interpolation is a long known technique for nonsymmetric FIR filter design that does not separate the response magnitude and delay in the approximation procedure [12]. Similarly, Savitzky-Golay filters [18] (which minimize a square error subject to a DC flatness constraint) can be generalized to nonsymmetric filter design with arbitrary non-integer DC delay [11]. Windows for the design of fractional-sample delay filters have also been studied [3]. Other recent work has given attention to the design of FIR filters that approximate, in the Chebyshev norm, a given response magnitude and group delay [4, 20] or a given response magnitude and phase [2, 10, 13, 14]. Baher [1] gives an analytic technique for obtaining nonsymmetric FIR filters having a maximally-flat behavior (see also [16, 17]). In [1] the degrees of flatness of the response magnitude and group delay at $\omega = 0$ can be different, but by no more than one. In this paper, an analytic technique is given for the same problem Baher addresses in [1], but the degree of flatness of the response magnitude and group delay need not be approximately equal.

The motivation for the design of such filters has all-ready been well expressed.

> ... the question arises as to whether it is possible to drop phase linearity outside the passband and utilise the available degrees of freedom to shape the amplitude response. [1]

> ... the exact linear phase and minimum phase solutions provide the extreme solutions: namely, the exact linear phase design unnecessarily constrains the linear phase response in the full frequency domain, while the minimum phase design drops the phase approximation altogether ... The goal is to design FIR filters whose properties are between those of exact linear and minimum phase filters. [13]

Like the above cited papers, this paper considers the problem of giving up exactly linear phase for approximately linear phase in exchange for a smaller delay and improvement in the response magnitude. This paper considers two-point Taylor approximation only. That is, derivatives of the realized response magnitude and group delay are made to vanish at $\omega = 0$ and $\omega = \pi$.

The approach taken in this paper is appropriate when:

1. Exactly linear phase is not required.
2. Some degree of phase linearity is desired.
3. A maximally-flat frequency response is desired.

The family of new FIR filters described below permits the degrees of freedom to be divided between the frequency response magnitude and the group delay in a way that is achieved by neither linear-phase nor square magnitude design methods. This family of filters is conveniently described by forming a right triangle (see Table 5) that extends infinitely along its hypotenuse and one of its legs. The set of linear-phase maximally-flat filters forms the hypotenuse, while the set of maximally-flat filters designed in the square magnitude domain forms the other leg. The filters introduced in this paper lie in the interior of the triangle. They make available trade-offs between the magnitude approximation and the group delay linearity provided by neither linear-phase nor square magnitude maximally-flat filters.

The nonsymmetric lowpass FIR filters discussed below are obtained by subjecting the frequency response magnitude and the group delay (individually) to differing numbers of flatness constraints. The equations that arise are nonlinear and have multiple solutions — some of which are extraneous (having unacceptable behavior in the interval $\omega \in (0, \pi)$ or being complex valued).

This paper identifies a class of filters which can be obtained via a straight forward use of linear algebra and polynomial root finding. For filters in this class, it is shown how to achieve a specified half-magnitude frequency and DC group delay (not necessarily integer). It is also found that by using filters in this class, the delay can be reduced while maintaining relatively constant group delay around $\omega = 0$, with no magnitude response degradation.

This paper is organized as follows. Sections 2 and 3 establish notation and describe the basic problem formulation. In Section 4, an example is presented and discussed. The example is described before the details are given to make clear the problem being addressed. Section 5 and Appendix A discuss how to obtain solutions, where the initial investigation of filter properties uses Gröbner bases. Section 6 and Appendix B explain how to modify the design technique to achieve a specified DC delay. Section 7 and Appendix C explain how to achieve both a specified DC delay and a specified cut-off frequency. An example of a longer filter is given in Section 8.

2 Notation

The transfer function of a length N FIR filter is denoted by $H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$. The real and imaginary parts of the frequency response are denoted by $R(\omega) = \Re\{H(e^{\jmath\omega})\} = \sum_{n=0}^{N-1} h(n)\cos n\omega$ and $I(\omega) = \Im\{H(e^{\jmath\omega})\} = -\sum_{n=0}^{N-1} h(n)\sin n\omega$. The frequency response square magnitude is then given by $F(\omega) = R^2(\omega) + I^2(\omega)$ and the group delay by $G(\omega) = (I(\omega)R'(\omega) - R(\omega)I'(\omega))/F(\omega)$.

The number of zeros of $H(z)$ at $z = -1$ is denoted by K. The zeros at $z = -1$ produce a flat behavior in the frequency response magnitude at $\omega = \pi$. Note that because $F(\omega)$ and $G(\omega)$ are even functions of ω, for odd l, $F^{(l)}(0)$ and $G^{(l)}(0)$ equal zero. The degree of flatness of the response magnitude at $\omega = 0$ is denoted by M:

$$F^{(2i)}(0) = 0 \qquad i = 1, \ldots, M. \tag{1}$$

The degree of flatness of the group delay at $\omega = 0$ is denoted by L:

$$G^{(2i)}(0) = 0 \qquad i = 1, \ldots, L. \tag{2}$$

A filter satisfying these conditions will be said to have flatness parameters (K, L, M). The DC constraint, $F(0) = 1$ is also needed.

For the filters described in the first part of this paper, $N = K + L + M + 1$. The only zeros lying on the unit circle are the K zeros at $z = -1$. The number of zeros not lying at $z = -1$ is then $L + M$. Hence, all the degrees of freedom available are used to attain degrees of flatness of the frequency response at $\omega = 0$ and $\omega = \pi$. Later on, specifications on the group delay at $\omega = 0$ and the half-magnitude frequency ω_o will be achieved by giving up one or two degrees of flatness in which case $N = K + L + M + 2$ or $N = K + L + M + 3$.

The moments of $H(z)$ will be useful. They are denoted and defined as:

$$m(k) = \sum_{n=0}^{N-1} n^k h(n). \tag{3}$$

The half-magnitude frequency is that frequency at which the response magnitude equals one half. Like the 3 dB point, it indicates the location of the cut-off point.

3 The Basic Problem Formulation

To obtain maximally-flat FIR filters having different degrees of magnitude and group delay flatness, the following mathematical problem formulation is suggested. Given the flatness parameters K, L, M, (with $K > 0$, $M \geq 0$, $L \leq M$), find N real filter coefficients $h(0), \ldots, h(N-1)$ such that:

1. $N = K + L + M + 1$.
2. $F(0) = 1$.
3. $H(z)$ has a root at $z = -1$ of order K.
4. $F^{(2i)}(0) = 0$ for $i = 1, \ldots, M$.
5. $G^{(2i)}(0) = 0$ for $i = 1, \ldots, L$.

Note that no constraints are made on the group delay at $\omega = \pi$ because the phase is unimportant in the stopband.

4 Example

To clarify the problem formulation above, an example is given before the design technique is described. Figures 1 and 2 illustrate four different FIR filters of length 13 for which $K+L+M=12$. Each of these filters has 6 zeros at $z=-1$ ($K=6$) and 6 zeros contributing to the flatness of the passband at $z=1$ ($L+M=6$). The four filters shown were obtained using the four values $L=0,1,2,3$. Several remarks are in order.

The filter having linear phase corresponds to the values $L=3$, $M=3$. This symmetric filter is given by the formulas for maximally flat filters Herrmann presents in [8].

The filter corresponding to the values $L=0$, $M=6$ can also be obtained using Herrmann's formulas by spectrally factoring a length 25 maximally-flat symmetric filter. For $L=0$, the group delay and impulse response shown in Figures 1 and 2 are those of the minimum phase spectral factor. Not counting time-reversals, there are four distinct real-valued impulse responses for this case, which are obtained by flipping zeros about the unit circle.

The other two filters shown in Figure 1 ($L=2$, $M=4$ and $L=1$, $M=5$) can not be obtained using the formulas of Herrmann. They are new and provide a compromise solution.

Just as there are multiple solutions for the $L=0$ case, there are multiple solutions in the cases for which $L>0$. Not counting time-reversal, for $L=1$, $M=5$, there are two distinct real solutions that posses monotonic response magnitudes. The monotonic $L=1$, $M=5$ real solution not shown in Figure 1 is shown in [19]. In addition, there is a real solution that is not monotonic, also shown in [19].

For $L=2$, $M=4$, there are two real solutions (not counting time-reversal) possessing non-monotonic response magnitudes. For $L=3$, $M=3$ there are three real solutions (not counting time-reversal) possessing non-monotonic response magnitudes. For $K=6$, $L+M=6$, $N=13$, all the real solutions are shown in [19]. Note that, while the four solutions for $L=0$, $M=6$ all have exactly the same frequency response magnitude, this is not true for $L>0$.

Our attention to passband monotonicity is motivated by the following. For the problem being addressed here, it is advantageous to know in advance that a frequency response magnitude is monotonic because this ensures that the behavior in $(0,\pi)$ is acceptable. If a solution is not monotonic, then the behavior in $(0,\pi)$ might be unacceptable due the extremely large potential errors [19].

Observe that for the filters shown in Figure 2, the way in which the passband zeros are split between the interior of the unit circle and its exterior is given by the values of L and M. It is interesting that for each K, L, and M, there is exactly one real solution possessing a monotonic magnitude having this property. Also, note that the location of these zeros in this regard was not part of the way in which the problem was formulated.

It may be observed that the half-magnitude frequencies of the four filters in Figure 2 are unequal. This is to be expected, because the half-magnitude frequency was not included in the problem formulation. In the problem formulation of Section 3 both the half-magnitude frequency and the DC group delay can be only indirectly controlled by specifying K, L, and M. Later on, achieving a specified group delay at $\omega = 0$ by giving up a degree of flatness will be described. The solutions shown in Figure 1 have approximately equal half-magnitude frequencies because they have the same number of zeros at $z = -1$. A set of solutions, the half-magnitudes of which are located over a greater interval, can be obtained by varying K. Figure 3 shows seven filters of length 13 obtained by varying K from 2 to 8.

The same filter length ($N = 13$) is used throughout most of the examples in this paper so that comparisons between them can be more easily made.

5 Obtaining Solutions

In terms of the moments of the filter, the square magnitude derivatives at $\omega = 0$ are given by:

$$F^{(2n)}(0) = \binom{2n}{n} m^2(n) + 2(-1)^n \sum_{i=0}^{n-1} \binom{2n}{i} (-1)^i m(i) m(2n-i) \tag{4}$$

where the i^{th} moment $m(i)$ is defined in (3). The general expressions for the derivatives of the group delay at $\omega = 0$ become very cumbersome, however, they can be simplified when the derivatives of the magnitude response $F(\omega)$ vanish at $\omega = 0$. When $F(0) = 1$ and $F^{(2i)}(0) = 0$ for $i = 1, \dots, n$, the group delay derivatives at $\omega = 0$ are given by:

$$G^{(2n)}(0) = (-1)^n \sum_{i=0}^{n} \frac{2n+1-2i}{2n+1} \binom{2n+1}{i} (-1)^i m(i) m(2n+1-i). \tag{5}$$

Notice that the expressions for $F^{(2n)}(0)$ and $G^{(2n)}(0)$ are nonlinear in $m(i)$ and hence nonlinear in $h(i)$. From (4), the first few derivatives of the square magnitude at $\omega = 0$ are:

$$F(0) = m_0^2 \tag{6}$$

$$F^{(2)}(0) = 2\, m_1^2 - 2\, m_0 m_2 \tag{7}$$

$$F^{(4)}(0) = 6\, m_2^2 + 2\, m_0 m_4 - 8\, m_1 m_3 \tag{8}$$

$$F^{(6)}(0) = 20\, m_3^2 - 2\, m_0 m_6 + 12\, m_1 m_5 - 30\, m_2 m_4. \tag{9}$$

From (5), the first few derivatives of the group delay at $\omega = 0$ are:

$$G(0) = m_0 m_1 \tag{10}$$

$$G^{(2)}(0) = -m_0 m_3 + m_1 m_2 \tag{11}$$

$$G^{(4)}(0) = m_0 m_5 - 3\, m_1 m_4 + 2\, m_2 m_3 \tag{12}$$

$$G^{(6)}(0) = -m_0 m_7 + 5\, m_1 m_6 - 9\, m_2 m_5 + 5\, m_3 m_4. \tag{13}$$

Note that $F^{(0)}(0) = m_0^2$. Hence the equation $F^{(0)}(0) = 1$ requires that m_0 equal 1 or -1. The solutions for which m_0 equals -1 are obtained by negating the solutions for which m_0 equals 1. Therefore, throughout this paper, $m_0 = 1$ is used. (Expression (10) becomes $G(0) = m_1$.) Also, throughout this paper, the variable A is used to represent the DC group delay, the group delay at $\omega = 0$ ($A = G(0) = m_1$).

Because the filter coefficients can be determined from sufficiently many of its moments, having enough moments permits one to obtain a solution. Since $m_0 = 1$, $m_1 = A$, from $F^{(2)}(0) = 0$ one gets $m_2 = A^2$, then using $G^{(2)}(0) = 0$, one gets $m_3 = A^3$, and so on. However, when $L < M - 2$, this process breaks down and prevents one from finding sufficiently many moments to compute a solution. If $L < M$, then it follows from (4) and (5) that $m(i) = A^i$ for $i = 0, \ldots, 2L+2$. If $L = M$, then it follows from (4) and (5) that $m(i) = A^i$ for $i = 0, \ldots, 2L+1$.

5.1 Using Gröbner Bases

One method of computing all solutions is provided by the Gröbner basis of the equation set. Given a system of multivariate polynomials, a Gröbner basis is a new set of multivariate polynomial equations, having the same set of solutions [5]. When the lexical ordering of monomials is used, and there is a finite number of solutions, the 'last' equation of the Gröbner basis will be a polynomial in a single variable — so its roots can be computed. These roots can be substituted into the remaining equations, etc — like back substitution in Gaussian elimination for linear equations. Unfortunately, Gröbner basis solutions are only practical for small problems, because computing a Gröbner basis is highly computationally intensive. Using the purely lexical ordering of monomials with $A < h(12) < h(11) < \cdots < h(0)$, the Gröbner basis for the case $K = 6$, $L = 2$, $M = 4$ was found using 'Singular' [7] and is given in Table 4. The solutions are given by the filter coefficients and DC group delay A for which all the expressions in Table 4 vanish.

The first observation is that there are 8 solutions for A: the 8 roots of the first polynomial in Table 4. Also note that each of the filter coefficients appears linearly in the remaining equations, hence, there are exactly 8 solutions to the problem formulated above. For the problem considered here, the general structure of the Gröbner basis is the same regardless of the actual values of K, L, and M: The first polynomial of the Gröbner basis is a univariate polynomial in A, the next N polynomials are linear in the filter coefficients $h(n)$. For a given K, L, M, the univariate polynomial in A will be called the *reference polynomial* below and denoted by $\mathcal{R}_{L,K,M}(A)$. The degree of $\mathcal{R}_{L,K,M}(A)$ is the number of solutions to the problem formulation of Section 3.

It should be noted that the time-reversal of each of the solutions h to the original problem must also be a solution, therefore, the roots of the reference polynomial must occur in pairs symmetrically around $(N-1)/2$. (If A_o is a root, then $N-1-A_o$ is also a root). In the example above, it is found that of the 8 roots of the reference polynomial, 6 are real. Each of the 6 consequent solutions appear in pairs with its time-reversal. Not counting time-reversals, there are 3 solutions to the problem. One of these solutions is monotonic and is shown in Figure 1.

5.2 The Number of Solutions

It is informative to form a table of the degrees of the reference polynomial for different K, L, M. It turns out that the degree of the reference polynomial is independent of K, the number of zeros at $z=-1$. The degree of the reference polynomial as a function of the remaining two parameters L and M is given in Table 5 for the first few L and M. Notice that

1. Along the diagonal $L = M$, the degree increases linearly with L and M: $degree = 2L+1 = 2M+1$.

2. Along the first column $L = 0$, the degree increases exponentially with M: $degree = 2^M$.

This suggests that the complexity required for computing the solutions when $L = M$ and when $L = 0$ must differ substantially. What happens for $0 < L < M$? How does the degree of the reference polynomial as a function of L and M change from exponential to linear?

Because the number of *real* solutions equals the number of *real* roots of the reference polynomial, this number is also of interest. Again, this number is independent of K. In Table 6, the number of real roots is shown for the first few values of L and M. Most relevant is the number of real *monotonic* solutions. This number is shown as a function of L and M in Table 7. It becomes clear from examining this table that the triangle can be split into two distinct regions.

Let us define two regions in the (L, M) plane. Define region I as all pairs L, M for which $\lfloor \frac{M-1}{2} \rfloor \leq L \leq M$. Define region II as all pairs (L, M) for which $0 \leq L \leq \lfloor \frac{M-1}{2} \rfloor - 1$. See Table 8. From Table 7, it can be seen that for each (L, M) in region I, there is one real solution having a monotonic response magnitude. From Table 7, for (L, M) in region II there are $2^{\lfloor \frac{M-1}{2} \rfloor}$ real solutions having a monotonic response magnitude. It turns out that for (L, M) in region I, all the variables in the problem formulation, except $G(0)$, are linearly related and can be eliminated, yielding a polynomial in $G(0)$, as detailed in the appendices.

An additional observation is worth noting. For a given L and M, how many of the $L + M$ 'passband' zeros lie inside the unit circle — and how many lie outside? It was found that for (L, M) in region I, the single real monotonic solution has L zeros outside the unit circle and M zeros inside. (If the time-reversal is used, then these numbers are interchanged). This is especially interesting because the location of the zeros in this regard was not part of the way in which the problem was formulated.

Although for each L and M in region II, there is more than one real solution having a monotonic response magnitude, it was found empirically that only one of these solutions has a monotonic group delay. Furthermore, of the real monotonic solutions in region II only that solution whose group delay is monotonic has the property that the zeros split between the exterior and interior of the unit circle according to L and M.

Computing the filters for L, M in region II quickly becomes intractable using Gröbner bases because the sizes of the polynomials and their coefficients grow exponentially with M. We were able to compute very few filters in region II using Gröbner bases — the complexity of that problem grows explosively.

In the problem formulation above, ω_o and $G(0)$ are determined by K, L, and M. But it is useful to know what values of ω_o and $G(0)$ can be achieved, and what the relationships among these parameters are. To this end, consider the following example in which filters of length 13 are examined. For each (K, L, M) with the property that (i) $K + L + M + 1 = 13 = N$ and (ii) (L, M) is in region I, there is a single real valued impulse response having a monotonic response magnitude. It is informative, and useful later on, to take $(\omega_o, G(0))$ as a pair of coordinates, and to plot these points in the plane. Figure 4 shows where these points lie for the length 13 filters in region I.

5.3 Region I Filters

It turns out that for (L, M) in region I, all the variables in the problem formulation above, except $G(0)$, are linearly related and can be eliminated, yielding a polynomial in $G(0)$. The details are described in Appendix A. By using the linear algebra and polynomial root finding described in Appendix A, the filters associated with region I can be designed without the explosive computational growth needed for Gröbner basis design of region II filters.

6 Specification of the DC Group Delay

For the ability to specify the value of the group delay at $\omega = 0$, it is necessary to give up one degree of flatness. In this case, the filter length N is related to the degrees of flatness by $N = K+L+M+2$.

Problem formulation: Given A (the desired group delay at $\omega = 0$), the integer K, and the sum $L+M$, (with $K > 0$, $L+M > 0$), find L, M, and N real filter coefficients $h(0), \ldots, h(N-1)$ such that:

1. $N = K + L + M + 2$.
2. $F(0) = 1$.
3. $G(0) = A$.
4. $H(z)$ has a root at $z = -1$ of order K.
5. $F^{(2i)}(0) = 0$ for $i = 1, \ldots, M$.
6. $G^{(2i)}(0) = 0$ for $i = 1, \ldots, L$.
7. $F(\omega)$ is monotonic in $(0, \pi)$.

A need not be an integer or even a rational number. The solutions in region I can be found by using the linear algebra described in Appendix B. The equations are roughly the same as those in Section 5.3, with slightly different matrix dimensions. The main difference is that no reference polynomial for A is needed. It should be noted that for this problem formulation there may not exist a solution having an acceptable response magnitude in $(0, \pi)$: The existence of an acceptable solution depends on the specified DC group delay. The design of filters in region II with a specified $G(0)$ are not addressed here.

6.1 Selecting L and M

Given $G(0)$, K, and $L+M$, obtaining a filter of length $N = K+L+M+2$ possessing a monotonic response magnitude requires selecting L and M appropriately. For an incorrect choice of L and M, the resulting response magnitude may not be monotonic, and may therefore have an unacceptable behavior between 0 and π.

To illustrate how to select L and M, consider again the filters shown in Figure 1, the DC group delays of which are tabulated in Table 2. Note that (i) for the case $K = 6$, $L = 2$, $M = 4$, the filter in Figure 1 has $G(0) = 4.32$, and (ii) for the case $K = 6$, $L = 1$, $M = 5$, the filter in Figure 1 has $G(0) = 2.83$. This suggests that a filter of the same length as those in Figure 1, having $G(0) \in [2.83, 4.32]$ and a monotonic response magnitude, can be obtained using $K = 6$, $L = 1$, $M = 4$. Then the total degree of flatness is one less than those of Figure 1 (11 instead of 12), but any DC group delay in the interval $[2.83, 4.32]$ can be achieved. Figure 5 shows a $K = 6$, $L = 1$, $M = 4$, $N = 13$ filter having a DC group delay of 3.2. For another example based on Figure 1, a filter having $G(0) \in [4.32, 6]$ can be obtained by using $K = 6$, $L = 2$, $M = 3$. Figure 5 shows such a filter having a DC group delay of 4.8.

As A, the specified DC group delay, is varied within the range $[4.32, 6]$, the half-magnitude frequency also varies. Note that ω_o is not part of the problem formulation above, so is determined by K, L, M, and A. However, it is informative to plot the way in which ω_o varies as A varies over $[4.32, 6]$, as is done in Figure 6. The filter in Figure 5 with $G(0) = 4.8$ is indicated by a circular mark in Figure 6.

Note that the endpoints of the curve in Figure 6 are the two points in Figure 4 indicated by $(6, 3, 3)$ and $(6, 2, 4)$. In general, when $L < M$, the endpoints of this curve are indicated by $(K, L+1, M)$ and $(K, L, M+1)$. When $L = M$, however, the upper endpoint of the curve is at $G(0) = (N-1)/2$.

From this discussion, it should be clear that, given $G(0)$ and K, the selection of L and M should be such that a curve like that of Figure 6 contains a point (ω_o, A) for which A is the specified DC group delay. That can be ascertained by using $(K, L+1, M)$ and $(K, L, M+1)$ in the problem formulation of Section 3 — and examining the DC group delay of the real monotonic solution in each case. (Note in addition, that this DC group delay is given by a root of a reference polynomial — so it can be found without computing the filter coefficients.)

The choice of K (the number of zeros at $z = -1$) should be made to approximately locate the transition region. Also, to obtain filters having $G(0) > (N-1)/2$, the time-reversal of solutions can be used. It is interesting to observe the behavior of the zero plot as the specified DC group delay is varied: When K is held constant and A is varied continuously towards $(N-1)/2$, zeros lying inside the unit circle sequentially travel along the real axis through $z = -1$ to its exterior.

7 Specification of Both the DC Group Delay and Half-Magnitude Frequency

This section, which describes how one achieves both a specified half-magnitude frequency and a specified DC group delay simultaneously, makes the approach taken in this paper more flexible and useful. (However, it should be noted that the existence of acceptable solutions does depend on the ω_o and $G(0)$ specified.) Achieving a specified ω_o and $G(0)$ is made possible by giving up two degrees of flatness. In this case, the filter length N is related to the flatness parameters by $N = K + L + M + 3$. Again, only filters in region I are considered here.

Problem formulation: Given the filter length N, A (the desired group delay at $\omega = 0$), and ω_o (the desired half-magnitude frequency), find K, L, M and the N real filter coefficients $h(0), \ldots, h(N-1)$ such that:

1. $N = K + L + M + 3$.
2. $F(0) = 1$.
3. $F(\omega_o) = \frac{1}{4}$.
4. $G(0) = A$.
5. $H(z)$ has a root at $z = -1$ of order K.
6. $F^{(2i)}(0) = 0$ for $i = 1, \ldots, M$.
7. $G^{(2i)}(0) = 0$ for $i = 1, \ldots, L$.
8. $F(\omega)$ is monotonic in $(0, \pi)$.

The region I solutions can be found using the linear algebra described in Appendix C.

7.1 Selection of K, L, M

Given a specified ω_o and $G(0)$, an important part of the procedure of Appendix C is the selection of K, L, and M. If the correct K, L, M are not chosen, then the solution given by the equations above may be unacceptable: It may be complex-valued and/or its frequency response may contain unacceptably large values.

To understand the relationship among K, L, M, ω_o, and $G(0)$, it is useful to construct a figure by plotting the way in which ω_o and $G(0)$ vary together, as was done in Figure 6. By plotting such curves for each pair, $(K, L+1, M)$ and $(K, L, M+1)$, in region I, it is found that a partitioning of a portion of the plane is obtained, as illustrated in Figure 7 for $N = 13$. Given a specified ω_o and $G(0)$, (in one of the sectors), this figure indicates exactly how K, L, and M should be selected so that a real monotonic solution is obtained that possesses the specified ω_o and $G(0)$ (as will be explained below). Several remarks are in order.

Any $(\omega_o, G(0))$ pair that falls in a sector in Figure 7 can be achieved. This makes fine tuning possible, which is especially useful for applications calling for fractional delay filters.

Notice that the nodes where the curves meet are those points shown in Figure 4. These are points where the maximal degree of flatness is attained ($N = K + L + M + 1$). Note also, that each two connected nodes in Figure 7 are also neighbors in the triangle shown in Table 5. For example, consider the point in Figure 7 indicated by $(7, 2, 3)$. Its two neighbors having greater $G(0)$, $(8, 2, 2)$ and $(6, 3, 3)$, lie above and to the right of $(7, 2, 3)$ in Table 5. (Note that the K coordinate is not used in Table 5.) The two neighbors of $(7, 2, 3)$ having smaller $G(0)$, $(8, 1, 3)$ and $(6, 2, 4)$, lie to left of and below $(7, 2, 3)$ in Table 5.

Recall from Section 6, that the curves in Figure 7 are points where $N = K + L + M + 2$: One degree of flatness is given up in order to achieve a specified $G(0)$ or a specified ω_o (but not both). In that case, it was found that if the endpoints of a curve segment have flatness parameters $(K, L+1, M)$ and $(K, L, M+1)$, then the points along the curve segment have flatness parameters (K, L, M).

Similarly, the flatness parameters for a point inside one of the sectors of Figure 7 is given by the minimum of the flatness parameters of the nodes at the vertices of the sector. Consider the four sided sectors in Figure 7, note that the flatness parameters of the four nodes at its vertices can be written as $(K + L, L + 1, M)$, $(K, L + 1, M)$, $(K + 1, L, M + 1)$, and $(K + 2, L, M)$. It follows that the interior points have flatness parameters (K, L, M).

Given a specified half-magnitude frequency ω_o and a specified DC group delay $G(0)$, the appropriate flatness parameters K, L, and M are selected by (i) locating the point $(\omega_o, G(0))$ in a sector in a figure such as Figure 7, and (ii) taking K, L, and M to be the minimum of the flatness parameters of the 3 or 4 encompassing nodes.

7.2 Example

In this example a set of filters of length 13 is designed. The filter responses are shown in Figure 8 and the flatness parameters used are tabulated in Table 10. Each filter achieves a specified half-magnitude frequency of 0.636π. The specified DC group delays are varied from 3.5 to 6 in steps of one half. Notice that all the response magnitudes are very similar to one another. Note that the group delay in the stopband behaves erratically, but recall that the phase in the stopband is considered to be of no importance. A conclusion that can be drawn from this example is that the group delay can be reduced, while retaining relatively linear phase around $\omega = 0$, with no magnitude response degradation.

7.3 Approximate Specification Sectors

As seen above, the selection of K, L, and M ensuring monotonicity requires a figure such as Figure 7 in which the specification sectors are shown. However, drawing these curves (although straight forward) requires more computation than is desirable for a convenient filter design program. Satisfactory specification sectors are obtained by connecting the nodes by linear line segments, as illustrated in Figure 9. The resulting diamond shapes are adequate approximations of the actual sectors and are easily obtained. The only points in the ω_o-$G(0)$ plane at which the approximate sectors provide different flatness parameters than do the exact sectors are near the boundaries of the sectors. For these ω_o-$G(0)$ coordinates, 'erroneously' selecting the flatness parameters of an adjacent sector leads only to slight non-monotonicity.

8 A Longer Example

To avoid giving the reader the impression that the approach taken in this paper is applicable only to filters of length 13, filters of length 23 are considered here. The ω_o of the previous example ($\omega_o = 0.636\pi$) is used and the DC group delay is varied from 7 to 11 in steps of one half. The filter responses are shown in Figure 10 and the flatness parameters used are tabulated in Table 11.

Notice in Figure 10 that in comparison to symmetric filters, the group delay can be reduced by just over a third (4/11) while maintaining a very flat group delay over the passband. In addition, the magnitudes of the filters with smaller delay have sharper transition regions (albeit, negligibly).

9 Conclusion

From the examples given in this paper, it is clear that the delay can be reduced while maintaining relatively constant group delay around $\omega = 0$, with no magnitude response degradation. That the problem of simultaneous magnitude and group delay approximation is a difficult one is made evident by the sophisticated iterative algorithms that are required [4, 13, 20]. However, as is shown in this paper, if a maximally-flat approximation is employed, then for a class of FIR filters there exists a straight forward design technique. This design technique (i) requires only the solutions to linear systems and the roots of polynomials, (ii) permits the specification of the half-magnitude frequency and the DC group delay, and (iii) produces filters having reduced delay (over symmetric filters) while maintaining a very flat group delay in the passband, with essentially identical magnitude.

It should be noted that even though the problem formulations described above require the solutions to nonlinear equations, a straight forward solution method is facilitated by the following: (i) Using moments $m(i)$ to obtain convenient expressions for $F^{(2i)}(0)$ and $G^{(2i)}(0)$. (ii) Identifying region I as a class of filters for which the formulated problem is more easily solved. (iii) Eliminating linearly dependent variables to obtain a polynomial in $G(0)$. (iv) Obtaining a tiling of part of the $\omega_o - G(0)$ plane by restricting attention to real monotonic solutions in region I.

It is expected that the filters discussed in this paper may find some of the same applications as the symmetric maximally-flat filters, for example [9, 21], and possibly some of the applications given in [12].

Appendices

A Calculating Filters in Region I

In this appendix, the problem formulated in Section 3 is solved for filters for which (L, M) lies in region I. Recall that the degree of $H(z)$ is $K+L+M$. First, the K zeros at $z = -1$ can be included in $H(z)$ explicitly by letting $H(z) = (z+1)^K P(z)$. $P(z)$ contains the $L+M$ zeros of $H(z)$ that contribute to the passband flatness ('P' is used here for 'passband'). The degree of $P(z)$ is $L+M$. Let $\mathbf{h}$ denote the vector of size N of coefficients of $H(z)$: $\mathbf{h} = [h(0), h(1), \ldots, h(N-1)]^t$, and let $\mathbf{p}$ denote the vector of size $L+M+1$ of coefficients of $P(z)$: $\mathbf{p} = [p(0), p(1), \ldots, p(L+M)]^t$. The two vectors $\mathbf{h}$ and $\mathbf{p}$ can be related by a convolution matrix T of size N by $(L+M+1)$: $\mathbf{h} = T\mathbf{p}$.

Let $\mathbf{m}$ denote the vector of the first $(L+M+1)$ moments: $\mathbf{m} = [m(0), m(1), \ldots, m(L+M)]^t$. Then the two vectors $\mathbf{m}$ and $\mathbf{h}$ can by related by a type of Vandermonde matrix Q of size $(L+M+1)$ by N: $\mathbf{m} = Q\mathbf{h}$, so that $\mathbf{m} = QT\mathbf{p}$. A solution can be written in terms of either the moments $\mathbf{m}$ or the filter coefficients $\mathbf{h}$. It turns out that solving the problem is facilitated by using the moments rather than the coefficients of $P(z)$, because the equations (4) and (5) are more conveniently written in terms of the moments than the filter coefficients. One also gets the two relationships, $\mathbf{p} = (QT)^{-1}\mathbf{m}$ and $\mathbf{h} = T(QT)^{-1}\mathbf{m}$.

The division into three cases facilitates the mathematics:

1. $L = M$ (the linear-phase diagonal), or $L = M-1$.
2. $L = M-2$.
3. The remaining filters in region I.

A.1 Case 1

Clearly, the monotonic linear-phase filters obtained when $L = M$ are more conveniently obtained via the formulas of Herrmann. The case $L = M$ is included here (i) for completeness and (ii) to verify that solutions to the problem formulation above (when $L = M$) are in fact the same as the filters of Herrmann.

The following discussion is valid for $L = M$ or $L = M-1$. Inspecting the equations $F^{(2i)}(0) = 0$ and $G^{(2i)}(0) = 0$ for $i = 0, \ldots, L$, it follows that $m(i) = A^i$ for $i = 0, 1, 2, \ldots, L+M+1$. Therefore $\mathbf{h}$ can be found in terms of A via the equation $\mathbf{h} = T(QT)^{-1}\mathbf{m}$.

To obtain the values A for which the flatness constraints are satisfied, consider the moment $m(L+M+1)$. The moment $m(L+M+1)$ is given by

$$m(L+M+1) = \sum_{n=0}^{N-1} n^{L+M+1} h(n) \tag{14}$$

$$= \mathbf{v}^t \mathbf{h} \tag{15}$$

$$= \mathbf{v}^t T (QT)^{-1} \mathbf{m} \tag{16}$$

where

$$\mathbf{v}^t = \left[0, 1, 2^{L+M+1}, \ldots, (N-1)^{L+M+1}\right]. \tag{17}$$

Since $m(L+M+1)$ also equals A^{L+M+1}, the reference polynomial for A is therefore

$$A^{L+M+1} - \mathbf{v}^t T(QT)^{-1}\mathbf{m}. \tag{18}$$

The filter coefficients are obtained via $\mathbf{h} = T(QT)^{-1}\mathbf{m}$.

For example, when $K = 6$, $L = 3$, $M = 3$, then

$$\mathbf{v}^t T(QT)^{-1} = \left[\begin{array}{ccccccc} 31185 & -\frac{173619}{2} & 74403 & -\frac{59003}{2} & 6195 & -\frac{1421}{2} & 42 \end{array}\right] \tag{19}$$

and the reference polynomial is

$$2\,A^7 - 84\,A^6 + 1421\,A^5 - 12390\,A^4 + 59003\,A^3 - 148806\,A^2 + 173619\,A - 62370. \tag{20}$$

As expected, $(N-1)/2 = 6$ is a root of this polynomial. The solution obtained using this root is the only solution having a monotonic response magnitude — and it is the same as that filter obtained using the formulas of Herrmann. For $L = M - 1$, the filters obtained here are not symmetric.

A.2 Case 2

Let $L = M - 2$. This case is similar to case 1. It follows from equations (4) and (5) that $m(i) = A^i$ for $i = 0, 1, 2, \ldots, L+M$. As above, $\mathbf{h}$ can be found in terms of A via $\mathbf{h} = T(QT)^{-1}\mathbf{m}$.

The equation $F^{(2M)}(0) = 0$ here simplifies to

$$(2M-1)A^{2M} - 2MA\,m(2M-1) + m(2M) = 0. \tag{21}$$

To obtain the values A for which the flatness constraints are satisfied, consider the two moments $m(2M-1)$ and $m(2M)$. They can be written in terms of $\mathbf{m}$.

$$\begin{aligned} m(i) &= \sum_{n=0}^{N-1} n^i h(n) && (22)\\ &= \mathbf{v}_i^t \mathbf{h} && (23)\\ &= \mathbf{v}_i^t T(QT)^{-1}\mathbf{m} && (24) \end{aligned}$$

for i equal to $2M-1$ and $2M$. The reference polynomial for A can now be found by substituting these two moment expressions in equation (21).

A.3 Case 3

In the following, suppose (L, M) lies in region I and $L < M - 2$. The approach in the following, is to write the moments in terms of A and to substitute these moment expressions into the equation $F^{(2M)}(0) = 0$ to obtain the reference polynomial in A.

Inspecting the equations $F^{(2i)}(0) = 0$ for $i = 0, \ldots, M$ and $G^{(2i)}(0) = 0$ for $i = 0, \ldots, L$, it follows that $m(i) = A^i$ for $i = 0, 1, 2, \ldots, 2L + 2$ when $M > L$. It will be useful to partition the vector $\mathbf{m}$ into two sub-vectors $\mathbf{m}_a$ and $\mathbf{m}_b$ of lengths $2L + 3$ and $M - L - 2$ respectively: $\mathbf{m} = \left[\mathbf{m}_a^t, \mathbf{m}_b^t\right]^t$ so that $\mathbf{m}_a = \left[1, A, A^2, \ldots, A^{2L+2}\right]^t$ and $\mathbf{m}_b$ represents the remaining moments whose dependence on A is to be determined: $\mathbf{m}_b = [m(2L + 3), m(2L + 4), \ldots, m(L + M)]^t$.

The next step is to obtain expressions for the moments $m(i)$ for i from $L + M + 1$ to $2M - 2$ in terms of the lower order moments. Expressions for these moments are needed because they appear in the equations $F^{(2i)}(0) = 0$ for $i = 0, \ldots, M - 1$. Expressions for these moments are obtained by using the definition (3). Let $\mathbf{m}_c$ be the vector of length $(M - L - 2)$ of these moments: $\mathbf{m}_c = [m(L + M + 1), m(L + M + 2), \ldots, m(2M - 2)]^t$. Then

$$\begin{aligned} \mathbf{m}_c &= V\mathbf{h} && (25)\\ &= VT(QT)^{-1}\mathbf{m} && (26)\\ &= U\mathbf{m} && (27) \end{aligned}$$

where V is a type of Vandermonde matrix of size $(M - L - 2)$ by N: $V_{i,j} = (j - 1)^{L+M+i}$ for $1 \le i \le M - L - 2$ and $1 \le j \le N$, and where U is defined as $VT(QT)^{-1}$. It will be useful to partition U into two sub-matrices: U_1 of size $(M - L - 2)$ by $(2L + 3)$, and U_2 of size $(M - L - 2)$ by $(M - L - 2)$ so that: $U = [U_1, U_2]$. Then

$$\mathbf{m}_c = U_1\mathbf{m}_a + U_2\mathbf{m}_b. \qquad (28)$$

Finally, looking closely at the equations $F^{(2i)}(0) = 0$ for $i = L + 2, \ldots, M - 1$, one finds that these equations can be written as

$$Y \begin{bmatrix} \mathbf{m}_b \\ \mathbf{m}_c \end{bmatrix} = \mathbf{r} \qquad (29)$$

where Y is the matrix of size $(M - L - 2)$ by $2(M - L - 2)$ having the following form

$$Y = \begin{bmatrix} -\binom{2L+4}{1}A & \binom{2L+4}{0} & & & & & & \\ -\binom{2L+6}{3}A^3 & \binom{2L+6}{2}A^2 & -\binom{2L+6}{1}A & \binom{2L+6}{0} & & & & \\ -\binom{2L+8}{5}A^5 & \binom{2L+8}{4}A^4 & -\binom{2L+8}{3}A^3 & \binom{2L+8}{2}A^2 & -\binom{2L+8}{1}A & \binom{2L+8}{0} & & \\ \vdots & & & & & & \ddots & \\ -\binom{2M-2}{c}A^c & & & \cdots & & & & \binom{2M-2}{0} \end{bmatrix} \qquad (30)$$

(with $c = 2(M - L) - 5$) and where $\mathbf{r}$ is the vector of length $M - L - 2$: $\mathbf{r} = [r_0, r_1, \ldots, r_{M-L-3}]^t$

where

$$r_k = A^{2L+4+2k} \sum_{i=0}^{1+2k} (-1)^i \binom{2L+2k+4}{i} \tag{31}$$

$$= -A^{2L+4+2k} \binom{2L+2k+3}{2k+1}. \tag{32}$$

The sum of binomial coefficients was simplified using a standard identity [6].

Define Y_1 and Y_2 to be two sub-matrices of Y, each of size $(M-L-2)$ by $(M-L-2)$ so that $Y = [Y_1 \;\; Y_2]$. Using the equations above, the sought moment vector $\mathbf{m}_b$ can be found as follows:

$$Y_1 \mathbf{m}_b + Y_2 \mathbf{m}_c = \mathbf{r} \tag{33}$$

$$Y_1 \mathbf{m}_b + Y_2 (U_1 \mathbf{m}_a + U_2 \mathbf{m}_b) = \mathbf{r} \tag{34}$$

$$(Y_1 + Y_2 U_2) \mathbf{m}_b = \mathbf{r} - Y_2 U_1 \mathbf{m}_a \tag{35}$$

$$\mathbf{m}_b = (Y_1 + Y_2 U_2)^{-1} (\mathbf{r} - Y_2 U_1 \mathbf{m}_a). \tag{36}$$

Note that the elements of the matrix $(Y_1 + Y_2 U_2)$ are polynomials in A (see the example below). A symbolic programming language, such as Maple, makes the computation of $\mathbf{m}_b$ via (36) straight forward, however.

To determine A, the reference polynomial in A is obtained by substituting the expressions obtained for the filter moments into the equation $F^{(2M)}(0) = 0$. A is taken to be one of the roots of the resulting univariate polynomial, and with this value of A, all the moments can be found using the expressions above. Finally, the filter coefficients $h(n)$ can be obtained by solving the linear system of equations $\mathbf{h} = T(QT)^{-1}\mathbf{m}$.

A.4 Root Selection

As mentioned above, the roots of the reference polynomial are located symmetrically around $(N-1)/2$ (because the time-reversal of a solution must also be a solution). (But when $M = L$, the reference polynomial has odd degree and the root at $(N-1)/2$ is simple.) Which real pair of roots produces the single monotonic response magnitude? It was found that in general the pair of roots lying closest to $(N-1)/2$ is the sought pair.

Since it is known that the reference polynomial is symmetric about $(N-1)/2$, before computing its roots, it is worthwhile translating it to obtain an even polynomial in A and reducing its degree by letting $A^2 = b$. The roots of the new polynomial can be mapped back, to obtain the roots of the reference polynomial.

A.5 Example

To clarify the expressions described above, this example is given for $K = 4$, $L = 2$, $M = 6$ ($N = 13$). For these values, $m(i) = A^i$ for $0 \le i \le 6$, $\mathbf{m}_a = \left[1, A, A^2, \ldots, A^6\right]^t$, $\mathbf{m}_b = [m_7, m_8]^t$, and

$$Y = \begin{bmatrix} -8\,A & 1 & 0 & 0 \\ -120\,A^3 & 45\,A^2 & -10\,A & 1 \end{bmatrix} \tag{37}$$

$$\mathbf{r} = \begin{bmatrix} -7\,A^8 \\ -84\,A^{10} \end{bmatrix} \tag{38}$$

$$U_1 = \begin{bmatrix} 155925 & -\frac{2584215}{2} & 2037618 & -1399985 & 521136 & -114744 & 15372 \\ \frac{18243225}{2} & -72741240 & \frac{224782353}{2} & -75495420 & 27293680 & -5765760 & 723912 \end{bmatrix} \tag{39}$$

$$U_2 = \begin{bmatrix} -1230 & 54 \\ -51480 & 1695 \end{bmatrix} \tag{40}$$

$$(Y_1 + Y_2 U_2)^{-1} = \begin{bmatrix} -\frac{3\,A^2 - 36\,A + 113}{16\,A^3 - 288\,A^2 + 1724\,A - 3432} & \frac{1}{240\,A^3 - 4320\,A^2 + 25860\,A - 51480} \\ -\frac{2\,A^3 - 205\,A + 858}{4\,A^3 - 72\,A^2 + 431\,A - 858} & \frac{2\,A}{60\,A^3 - 1080\,A^2 + 6465\,A - 12870} \end{bmatrix} \tag{41}$$

$$\mathbf{m}_b = \begin{bmatrix} m(7) \\ m(8) \end{bmatrix} = (Y_1 + Y_2 U_2)^{-1}(\mathbf{r} - Y_2 U_1 \mathbf{m}_a) \tag{42}$$

This gives:

$$m(7) = \frac{154\,A^{10} - 2520\,A^9 + 7910\,A^8 + 102480\,A^7 - 1247568\,A^6 + 7318080\,A^5 - 27529020\,A^4 + 63914400\,A^3 - 83541501\,A^2 + 49533660\,A - 6081075}{160\,A^3 - 2880\,A^2 + 17240\,A - 34320} \tag{43}$$

$$m(8) = \frac{14\,A^{11} - 7175\,A^9 + 132510\,A^8 - 1247568\,A^7 + 7318080\,A^6 - 27529020\,A^5 + 63914400\,A^4 - 83541501\,A^3 + 49533660\,A^2 - 6081075\,A}{20\,A^3 - 360\,A^2 + 2155\,A - 4290} \tag{44}$$

Substituting these expressions for $m(i)$ into the equation $F^{(12)}(0) = 0$, one obtains the reference polynomial $\mathcal{R}_{4,2,6}(A)$:

$$4\,A^{12} - 288\,A^{11} + 9260\,A^{10} - 175440\,A^9 + 2175162\,A^8 - 18519264\,A^7 + 110406800\,A^6 - 460618560\,A^5 + 1317973984\,A^4 - 2469896448\,A^3 + 2765955690\,A^2 - 1514878200\,A + 212837625 = 0. \tag{45}$$

By making the change of variable, $A = \sqrt{d} + 6$, the following lower order polynomial is obtained.

$$4\,d^6 - 244\,d^5 + 4362\,d^4 - 23632\,d^3 + 28384\,d^2 - 33174\,d - 46575 = 0. \tag{46}$$

The roots of the lower order polynomial are:

$$\{0.86977247 \pm 1.6629531i, -0.70219452, 7.3807806, 18.989484, 33.592385\}. \tag{47}$$

Each of the real positive roots of the reduced polynomial are mapped to a pair of real roots of the reference polynomial located symmetrically around $(N-1)/2 = 6$. Hence, the real roots of the reference polynomial are:

$$\{0.20410624, 1.6423074, 3.2832408, 8.7167592, 10.357693, 11.795894\}. \tag{48}$$

The pair of roots giving rise to the single monotonic response magnitude is that pair lying closest to 6, namely, $A = 3.2832408$ and $A = 8.7167592$. The solution obtained using one of these two values is the time-reversal of the solution obtained using the other. In practice, it generally make senses to employ the solution having smaller $G(0)$.

B Calculating Filters with Specified DC Group Delay

In this appendix, the problem formulated in Section 6 is solved for filters for which (L, M) lies in region I. Taking $N = K + L + M + 2$, the degree of $H(z)$ is $K + L + M + 1$ and, as above, the K zeros at $z = -1$ can be included in $H(z)$ by letting $H(z) = (z+1)^K P(z)$ where the degree of $P(z)$ is $L + M + 1$. Let $\mathbf{h} = [h(0), \ldots, h(N-1)]^t$, $\mathbf{p} = [p(0), \ldots, p(L+M+1)]^t$. Let T be the convolution matrix of size N by $(L+M+2)$ so that $\mathbf{h} = T\mathbf{p}$. Let $\mathbf{m}$ denote the vector of the first $L + M + 2$ moments, $\mathbf{m} = [m(0), \ldots, m(L+M+1)]^t$. Then $\mathbf{m} = Q\mathbf{h}$ where Q is the appropriate Vandermonde-like matrix of size $(L+M+2)$ by N. As in the previous section, one has $\mathbf{p} = (QT)^{-1}\mathbf{m}$ and $\mathbf{h} = T(QT)^{-1}\mathbf{m}$.

It is necessary to consider two cases:

1. $L = M$ or $L = M - 1$.

2. The remaining filters in region I.

B.1 Case 1

Suppose $A = G(0)$ is given and $L = M$ or $L = M - 1$. As above, $m(i) = A^i$ for $i = 0, 1, 2, \ldots, L + M + 1$, therefore, $\mathbf{h}$ can be obtained directly from $\mathbf{h} = T(QT)^{-1}\mathbf{m}$.

B.2 Case 2

Proceed as in Section A.3, but use $\mathbf{m}_b = [m(2L+3), \ldots, m(L+M+1)]^t$ and $\mathbf{m}_c = [m(L+M+2), \ldots, m(2M)]^t$. The modified matrix V is of size $(M-L-1)$ by N and has entries $V_{i,j} = (j-1)^{L+M+1+i}$ for $1 \leq i \leq M-L-1$ and $1 \leq j \leq N$. U, U_1, U_2, Y, Y_1 and Y_2 are all given by the same expressions as in Section A.3 but their new dimensions are as follows: U is of size $(M-L-1)$ by $(L+M+2)$. U_1 is of size $(M-L-1)$ by $(2L+3)$. U_2 is of size $(M-L-1)$ by $(M-L-1)$. Y is of size $(M-L-1)$ by $2(M-L-1)$. Y_1 and Y_2 are of size $(M-L-1)$ by $(M-L-1)$. $\mathbf{r}$ is of length $(M-L-1)$.

C Calculating Filters with Both Specified DC Group Delay and Half-Magnitude Frequency

In this appendix, the problem formulated in Section 7 is solved for filters for which (L, M) lies in region I. Taking $N = K+L+M+3$, the degree of $H(z)$ is $K+L+M+2$ and, as above, the K zeros at $z=-1$ can be included in $H(z)$ by letting $H(z) = (z+1)^K P(z)$ where the degree of $P(z)$ is $L+M+2$. Let $\mathbf{h} = [h(0), \ldots, h(N-1)]^t$, $\mathbf{p} = [p(0), \ldots, p(L+M+2)]^t$. Let T be the convolution matrix of size N by $(L+M+3)$ so that $\mathbf{h} = T\mathbf{p}$. Let $\mathbf{m}$ denote the vector of the first $L+M+3$ moments, $\mathbf{m} = [m(0), \ldots, m(L+M+2)]^t$. Then $\mathbf{m} = Q\mathbf{h}$ where Q is the appropriate type of Vandermonde matrix of size $(L+M+3)$ by N. As above, one has $\mathbf{p} = (QT)^{-1}\mathbf{m}$ and $\mathbf{h} = T(QT)^{-1}\mathbf{m}$.

It is necessary to consider three cases:

1. $L = M$ or $L = M-1$.
2. $L = M-2$.
3. The remaining filters in region I.

C.1 Case 1

Suppose $A = G(0)$ and ω_o are given and $L = M$ or $L = M-1$. As above, $m(i) = A^i$ for $i = 0, \ldots, L+M+1$. Let α denote the moment $m(L+M+2)$. Then $\mathbf{m} = [1, A, A^2, \ldots, A^{L+M+1}, \alpha]^t$. One obtains $\mathbf{h}$ as a linear function of α via $\mathbf{h} = T(QT)^{-1}\mathbf{m}$. The equation $F(\omega_o) = \frac{1}{4}$ yields a quadratic polynomial equation for α, so α can be found via the quadratic formula.

C.2 Case 2

Suppose $A = G(0)$ and ω_o are given and $L = M-2$. As in Section A.2, $m(i) = A^i$ for $i = 0, \ldots, 2L+1$. Let α denote the moment $m(2L+3)$. In this case $\mathbf{m} = [1, A, A^2, \ldots, A^{2L+2}, \alpha, m(2M)]^t$. Note that (21) applies here, so one obtains

$$m(2M) = 2MA\alpha - (2M-1)A^{2M}. \tag{49}$$

Again, one obtains $\mathbf{h}$ as a linear function of α via $\mathbf{h} = T(QT)^{-1}\mathbf{m}$, and the equation $F(\omega_o) = \frac{1}{4}$ yields a quadratic polynomial equation for α.

C.3 Case 3

Suppose $A = G(0)$ and ω_o are given and L, M are in region I with $L < M-2$. The following discussion is similar to that of A.3. Partition $\mathbf{m}$ so that

$$\mathbf{m} = \begin{bmatrix} \mathbf{m}_a \\ \alpha \\ \mathbf{m}_b \end{bmatrix} \tag{50}$$

where $\mathbf{m}_a = [1, \ldots, A^{2L+2}]^t$, α denotes the moment $m(2L+3)$, and $\mathbf{m}_b = [m(2L+4), \ldots, m(L+M+2)]^t$. Also, define $\mathbf{m}_c = [m(L+M+3), \ldots, m(2M)]^t$. Then $\mathbf{m}_c = U\mathbf{m}$, as in Section A.3, where $U = VT(QT)^{-1}$ and V is a type of Vandermonde matrix of size $(M-L-2)$ by N: $V_{i,j} = (j-1)^{L+M+2+i}$ for $1 \leq i \leq M-L-2$ and $1 \leq j \leq N$. Partition U into 3 sub-matrices so that $U = [U_1 \;\; \mathbf{s} \;\; U_2]$ where U_1 is of size $(M-L-2)$ by $(2L+3)$, where $\mathbf{s}$ is a column vector, and U_2 is of size $(M-L-2)$ by $(M-L-1)$. Then

$$\mathbf{m}_c = U_1\mathbf{m}_a + U_2\mathbf{m}_b + \alpha\mathbf{s}. \tag{51}$$

As is done in Section A.3, the equations $F^{(2i)}(0) = 0$ for $i = L+2, \ldots, M$ can be written as

$$Y \begin{bmatrix} \alpha \\ \mathbf{m}_b \\ \mathbf{m}_c \end{bmatrix} = \mathbf{r} \tag{52}$$

where Y has the form given in (30) but is of size $(M-L-1)$ by $2(M-L-1)$, and $\mathbf{r}$ has the form given in (31) but is of length $(M-L-1)$. Partition Y into 3 sub-matrices so that $Y = [\mathbf{g} \;\; Y_1 \;\; Y_2]$ where $\mathbf{g}$ is a column vector, Y_1 is a square matrix, and Y_2 is a matrix of size $(M-L-1)$ by $(M-L-2)$. With these expressions, one obtains $\mathbf{m}_b$ as follows:

$$\alpha\mathbf{g} + Y_1\mathbf{m}_b + Y_2\mathbf{m}_c = \mathbf{r} \tag{53}$$

$$\alpha\mathbf{g} + Y_1\mathbf{m}_b + Y_2(U_1\mathbf{m}_a + U_2\mathbf{m}_b + \alpha\mathbf{s}) = \mathbf{r} \tag{54}$$

$$(Y_1 + Y_2U_2)\mathbf{m}_b = \mathbf{r} - Y_2U_1\mathbf{m}_a - \alpha(\mathbf{g} + Y_2\mathbf{s}) \tag{55}$$

$$\mathbf{m}_b = (Y_1 + Y_2U_2)^{-1}(\mathbf{r} - Y_2U_1\mathbf{m}_a - \alpha(\mathbf{g} + Y_2\mathbf{s})). \tag{56}$$

This gives $\mathbf{m}$ as a linear function of α, and in turn, gives $\mathbf{h}$ as a linear function of α. The equation $F(\omega) = \frac{1}{4}$ yields a quadratic polynomial equation for α.

C.4 Root Selection

The polynomial for α above is second degree in each case. It was found that usually one of the roots gives a solution with an unacceptable response. Always one of the roots will give a solution with a monotonic response, however. That solution will be the one having $M+1$ zeros outside the unit circle, and $L+1$ zeros inside the unit circle.

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Table 1: Notation.

Functions	
$F(\omega)$	magnitude squared
$G(\omega)$	group delay
Parameters	
N	filter length
K	number of zeros at $z = -1$
A	group delay at $\omega = 0$
ω_o	half-magnitude frequency
Flatness	
$K + L + M$	total degrees of flatness
K	magnitude flatness at $\omega = \pi$
L	group delay flatness at $\omega = 0$
M	magnitude flatness at $\omega = 0$

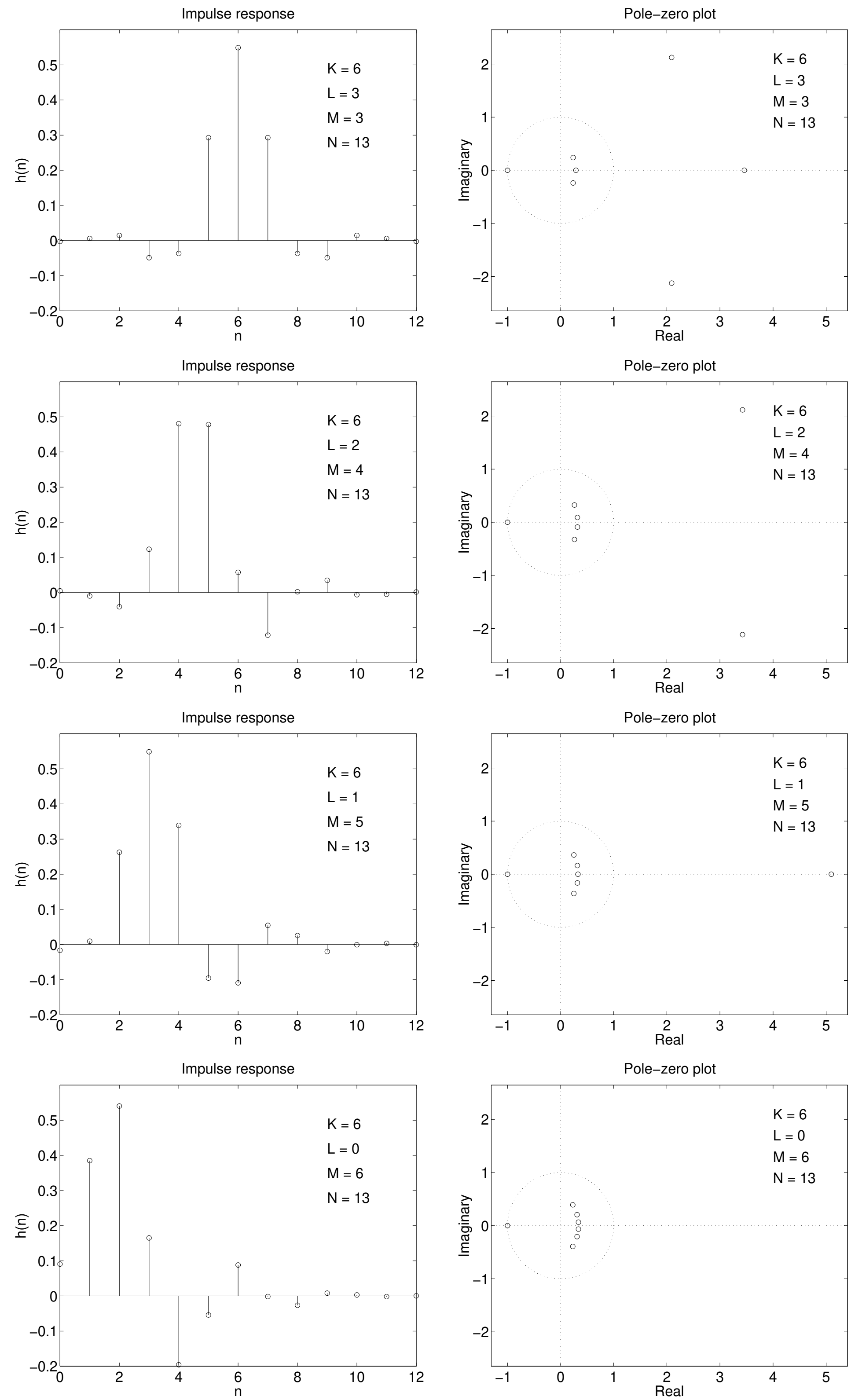

Figure 1: A selection of nonsymmetric maximally-flat filters of length 13. The four filters shown are solutions to the problem formulation of Section 3. For each filter shown, the zero at $z = -1$ is of multiplicity 6.

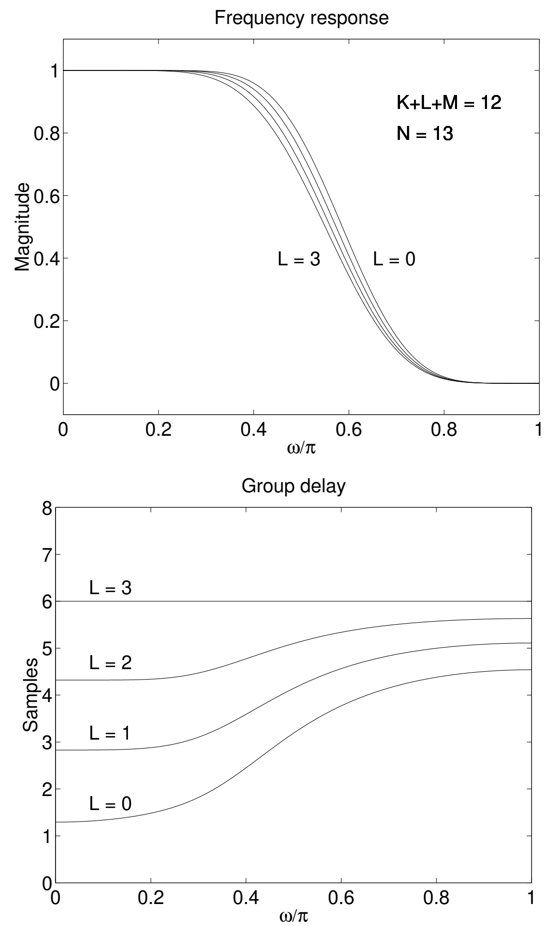

Figure 2: The magnitude responses and group delays of the filters shown in Figure 1. See Table 2.

Table 2: The DC group delays and half-magnitude frequencies of the filters shown in Figure 1. $L+M$ is the number of zeros contributing to the passband.

N	K	L	M	ω_o/π	$G(0)$
13	6	3	3	0.5502	6
		2	4	0.5613	4.3201
		1	5	0.5731	2.8290
		0	6	0.5862	1.2934

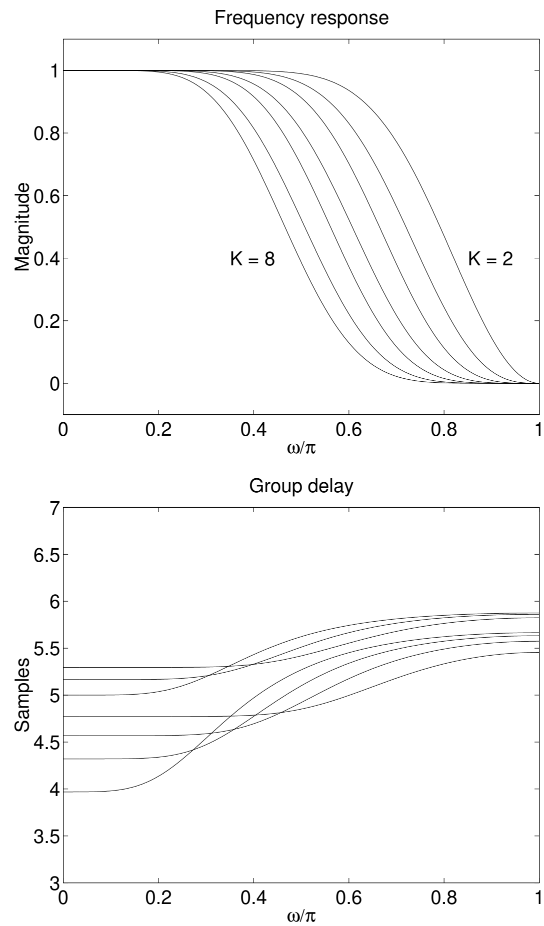

Figure 3: $N = 13 = K + L + M + 1$. See Table 3.

Table 3: The DC group delays and half-magnitude frequencies of the filters shown in Figure 3.

N	K	L	M	ω_o/π	$G(0)$
13	2	4	6	0.7915	4.771131
	3	4	5	0.7197	5.294824
	4	3	5	0.6642	4.567345
	5	3	4	0.6070	5.1654
	6	2	4	0.5613	4.3201
	7	2	3	0.5073	5
	8	1	3	0.4656	3.9683

Table 4: The Gröbner basis for the case $K = 6$, $L = 2$, $M = 4$, $N = 13$.

$2\,A^8 - 96\,A^7 + 1928\,A^6 - 21024\,A^5 + 134858\,A^4 - 513264\,A^3 + 1102872\,A^2 - 1162656\,A + 405405$
$92160\,h(12) - 2\,A^6 + 66\,A^5 - 845\,A^4 + 5280\,A^3 - 16523\,A^2 + 23364\,A - 10395$
$7680\,h(11) - A^5 + 25\,A^4 - 230\,A^3 + 950\,A^2 - 1689\,A + 945$
$15360\,h(10) + 2\,A^6 - 68\,A^5 + 885\,A^4 - 5560\,A^3 + 17323\,A^2 - 24132\,A + 10395$
$1536\,h(9) + A^5 - 27\,A^4 + 262\,A^3 - 1122\,A^2 + 2041\,A - 1155$
$6144\,h(8) - 2\,A^6 + 70\,A^5 - 933\,A^4 + 5936\,A^3 - 18475\,A^2 + 25284\,A - 10395$
$768\,h(7) - A^5 + 29\,A^4 - 302\,A^3 + 1366\,A^2 - 2577\,A + 1485$
$4608\,h(6) + 2\,A^6 - 72\,A^5 + 989\,A^4 - 6456\,A^3 + 20267\,A^2 - 27204\,A + 10395$
$768\,h(5) + A^5 - 31\,A^4 + 350\,A^3 - 1730\,A^2 + 3489\,A - 2079$
$6144\,h(4) - 2\,A^6 + 74\,A^5 - 1053\,A^4 + 7168\,A^3 - 23371\,A^2 + 31044\,A - 10395$
$1536\,h(3) - A^5 + 33\,A^4 - 406\,A^3 + 2262\,A^2 - 5353\,A + 3465$
$15360\,h(2) + 2\,A^6 - 76\,A^5 + 1125\,A^4 - 8120\,A^3 + 28843\,A^2 - 42564\,A + 10395$
$7680\,h(1) + A^5 - 35\,A^4 + 470\,A^3 - 3010\,A^2 + 9129\,A - 10395$
$92160\,h(0) - 2\,A^6 + 78\,A^5 - 1205\,A^4 + 9360\,A^3 - 38123\,A^2 + 75972\,A - 56475$

Table 5: The degree of the reference polynomial $\mathcal{R}_{K,L,M}(A)$ as a function of L and M.

		L							
		0	1	2	3	4	5	6	7
M	0	1							
	1	2	3						
	2	4	4	5					
	3	8	6	6	7				
	4	16	8	8	8	9			
	5	32	16	10	10	10	11		
	6	64	26	12	12	12	12	13	
	7	128	48	24	14	14	14	14	15

Table 6: The number of real roots of the reference polynomial $\mathcal{R}_{K,L,M}(A)$.

		L							
		0	1	2	3	4	5	6	7
M	0	1							
	1	2	3						
	2	2	4	5					
	3	4	4	6	7				
	4	4	4	6	8	9			
	5	8	6	6	8	10	11		
	6	8	8	6	8	10	12	13	
	7	16	8	8	8	10	12	14	15

Table 7: The number of real monotonic solutions, not counting time-reversals.

		L							
		0	1	2	3	4	5	6	7
M	0	1							
	1	1	1						
	2	1	1	1					
	3	2	1	1	1				
	4	2	1	1	1	1			
	5	4	2	1	1	1	1		
	6	4	2	1	1	1	1	1	
	7	8	4	2	1	1	1	1	1

Table 8: Regions I and II.

		L											
		0	1	2	3	4	5	6	7	8	9	10	$\cdots$
M	0												
	1												
	2												
	3												
	4												
	5												
	6												
	7												
	8												
	9							region I					
	10		region II										
	$\vdots$					$\ddots$							$\ddots$

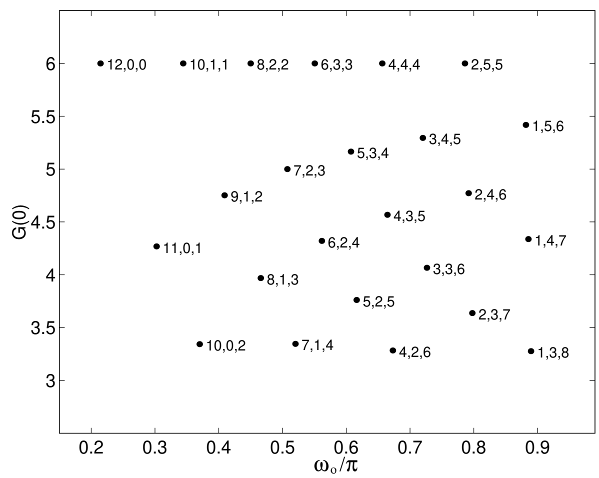

Figure 4: Locations in the ω_o-$G(0)$ plane of each of the length 13 filters in region I for which $N = K + L + M + 1$. The horizontal axis indicates the half-magnitude frequency. The flatness parameters (K, L, M) are indicated.

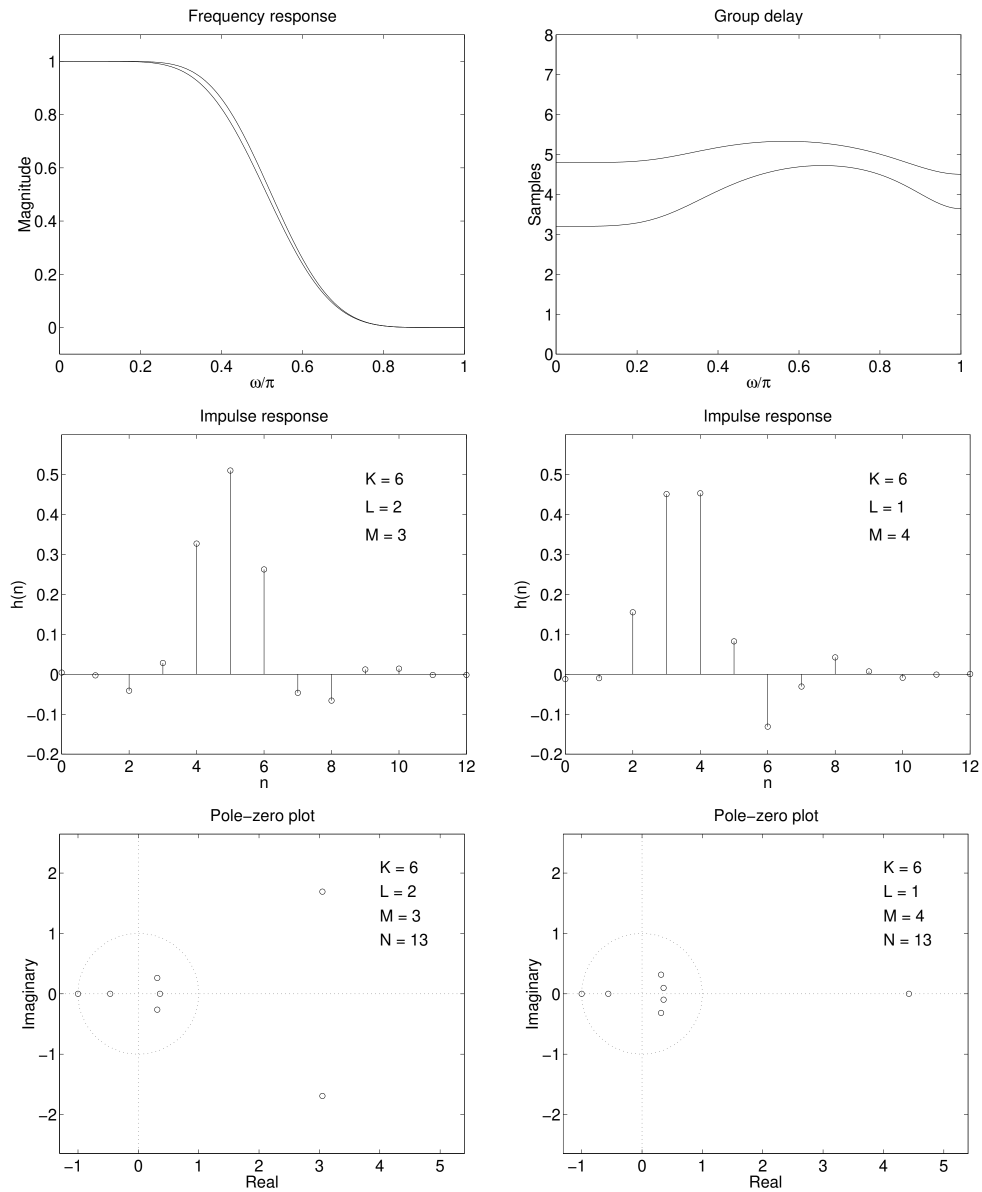

Figure 5: $K = 6$, $L + M = 6$, $N = 13$. Specified DC group delay: $G(0) = 4.8$ and $G(0) = 3.2$. See Table 9. For both filters, the number of zeros at $z = -1$ is 6.

Table 9: The DC group delays and half-magnitude frequencies of the filters shown in Figure 5.

N	K	L	M	ω_o/π	$G(0)$
13	6	1	4	0.5235	3.2
		2	3	0.5124	4.8

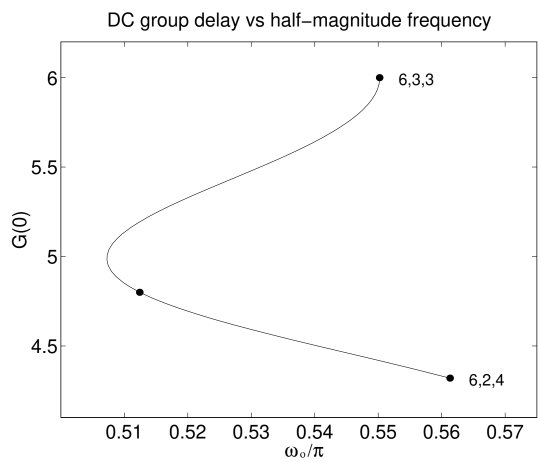

Figure 6: $K = 6$, $L = 2$, $M = 3$, $N = 13$. The curve describes the way in which the half-magnitude frequency varies as the specified DC group delay varies.

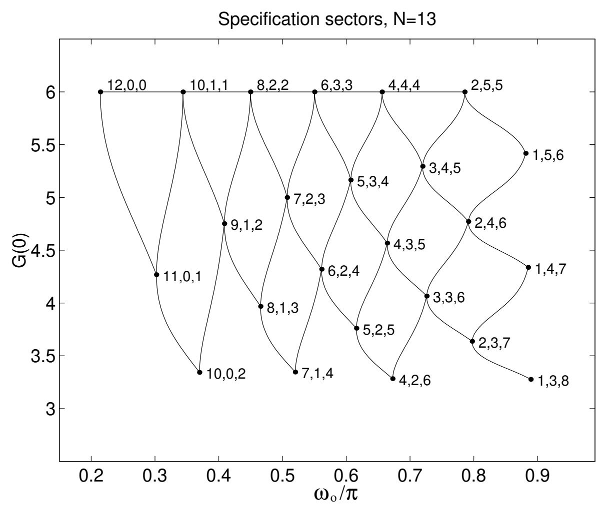

Figure 7: Specification sectors in the ω_o-$G(0)$ plane for length 13 filters in region I. The nodes are points at which $N = K+L+M+1$. The three integers by each point give K, L and M respectively.

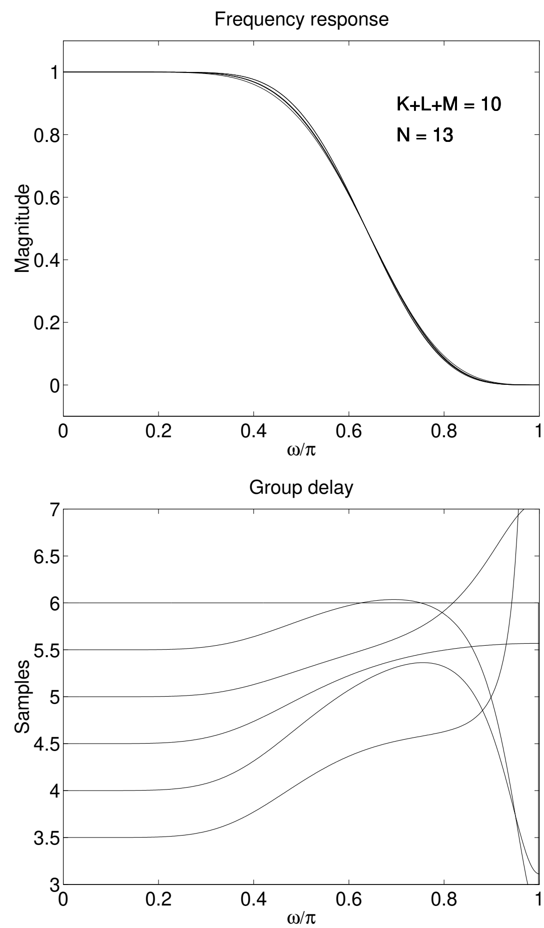

Figure 8: $N = 13 = K + L + M + 3$. Specified half-magnitude frequency and DC group delay. See Table 10.

Table 10: The flatness parameters of the filters shown in Figure 8.

N	ω_o/π	$G(0)$	K	L	M
13	0.636	3.5	3	2	5
		4	3	2	5
		4.5	4	2	4
		5	3	3	4
		5.5	3	3	4
		6	4	3	3

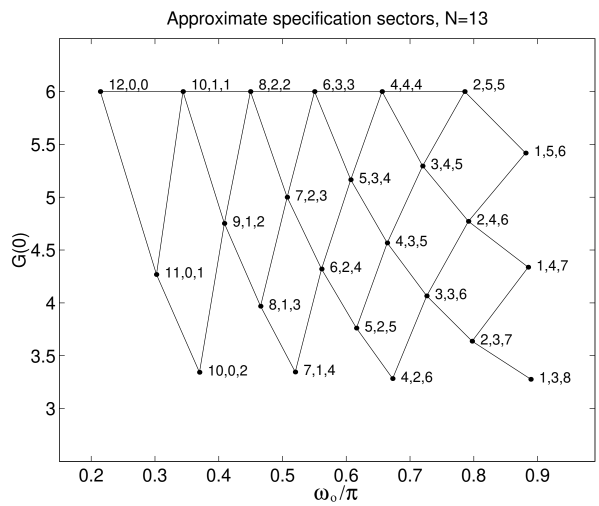

Figure 9: Approximation of the specification sectors shown in Figure 7.

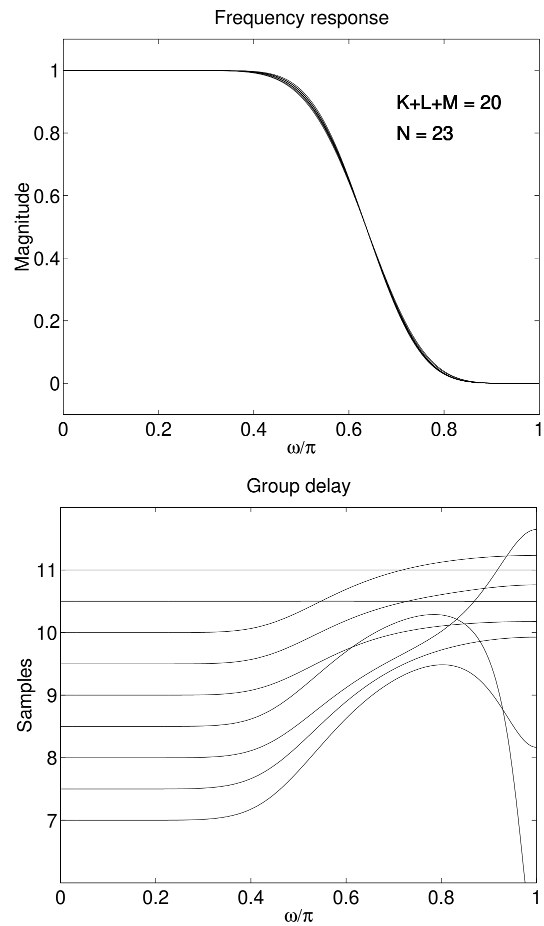

Figure 10: $N = 23 = K + L + M + 3$. Specified half-magnitude frequency and DC group delay. See Table 11. The response magnitudes are essentially the same, but the filters having smaller delay have very slightly steeper transition region responses.

Table 11: The flatness parameters of the filters shown in Figure 10.

N	ω_o/π	$G(0)$	K	L	M
23	0.636	7	6	4	10
		7.5	7	4	9
		8	6	5	9
		8.5	6	5	9
		9	7	5	8
		9.5	6	6	8
		10	7	6	7
		10.5	7	6	7
		11	6	7	7