Narrow-Band Low-Pass Digital Differentiator Design

Ivan Selesnick
Polytechnic University
Brooklyn, New York

selesi@poly.edu
http://taco.poly.edu/selesi
The frequency response of the ideal lowpass digital differentiator is

\[ H_{LP}(e^{j\omega}) = \begin{cases} 
  j \omega & |\omega| < \omega_c \\
  0 & \omega_c < |\omega| < \pi 
\end{cases} \] (1)

- It is a narrow-band filter if \( \omega_c \) is much smaller than \( \pi \).
- A narrow-band filter should have a long impulse response.
- \( \Rightarrow \) It is desirable to have simple design algorithms so that ill-conditioning and computational complexity is minimized.
- The window method for FIR filter design is a natural choice in this case. The design method described here gives an alternative approach.
The next slide illustrates the result of filtering an EOG signal with:

1. a full-band differentiator and

2. a narrow-band lowpass differentiator

Differentiation with the full-band differentiator yields an extremely noisy signal, while lowpass differentiation gives a more useful result.
Design

To avoid the undesirable amplification of noise in digital differentiation, lowpass differentiators can be used in place of full-band ones.

Low-pass digital differentiator design:

1. Maxflat
2. Least-squares
3. Remez
4. Flat passband, Equiripple stopbands (Kaiser, Rabiner, Vaidyanathan)
Our Approach

We describes a simple formulation for the non-iterative design of narrow-band FIR linear-phase lowpass digital differentiators.

- The filters are flat around dc and have equally spaced nulls in the stopband.
- The impulse response can be written as a sum of sines (Frequency sampling expression).
- The design problem is formulated so as to avoid the complexity or ill-conditioning of standard methods for the design of similar filters when those methods are used to design narrow-band filters with long impulse responses.
The sinc function is given by

\[ \text{sinc}(f) := \frac{\sin(\pi f)}{\pi f}. \]

The function \( \text{sinc}(f) \) is symmetric (\( \text{sinc}(-f) = \text{sinc}(f) \)) and equal to zero for \( f = \pm 1, \pm 2, \pm 3, \ldots \); therefore if we define

\[ s_k(f) := \text{sinc}(f - k) - \text{sinc}(f + k) \]

then we have

1. \( s_k(f) \) is antisymmetric, \( \text{sinc}(-f) = -\text{sinc}(f) \).

2. \( s_k(f) = 0 \) for \( f \in \mathbb{Z}/\{\pm k\} \).

\( (s_k(f) = 0 \text{ whenever } f \text{ is an integer different from } \pm k. \)
Analog Lowpass Differentiator

The digital filter design procedure we propose begins with an analog frequency response having the following form:

\[ A(f) = \sum_{k=1}^{K} a(k, K) \left( \text{sinc}(f - k) - \text{sinc}(f + k) \right) \]

Therefore, the frequency response \( A(f) \) has the following properties:

1. \( A(f) \) is antisymmetric, \( A(-f) = -A(f) \).
2. \( A(f) = 0 \) for \( f = 0 \), and for \( f = \pm(K + 1), \pm(K + 2), \pm(K + 3), \ldots \).

- The frequency response \( A(f) \) is zero at \( f = 0 \).
- The first null in the stopband depends on \( K \).
- The exact behavior of \( A(f) \) depends on the coefficients \( a(k, K) \), however, the uniformly spaced nulls in the stopband ensures that the attenuation increases with frequency.
Problem Formulation

The coefficients $a(k, K)$ are to be determined so that the frequency response $A(f)$ approximates $f$ near $f = 0$

$$A(f) \approx f.$$

Given $K$, find $a(k, K)$ for $1 \leq k \leq K$ such that the derivatives of $A(f)$ at $f = 0$ match the derivatives of the ideal differentiator $\text{IdealDiff}(f) := f$ at the point $f = 0$:

$$A^{(1)}(0) = 1 \quad (2)$$
$$A^{(i)}(0) = 0, \quad i = 3, 5, \ldots, 2K - 1. \quad (3)$$

- The even derivatives are automatically zero because $A(f)$ is an odd function, $A(-f) = -A(f)$.
- This is a linear system of equations with an equal number of equations and variables.
- The stopband of $A(f)$ is neither equiripple nor maximally flat.
Example

For example, when $K = 1$, we have

$$a(1, 1) = \frac{1}{2}.$$ 

When $K = 2$, we have

$$a(1, 2) = -\frac{1}{6} + \frac{1}{9} \pi^2$$
$$a(2, 2) = -\frac{4}{3} + \frac{2}{9} \pi^2$$

When $K = 3$, we have

$$a(1, 3) = \frac{1}{48} - \frac{13}{288} \pi^2 + \frac{7}{480} \pi^4$$
$$a(2, 3) = \frac{16}{15} - \frac{16}{9} \pi^2 + \frac{14}{75} \pi^4$$
$$a(3, 3) = \frac{243}{80} - \frac{81}{32} \pi^2 + \frac{189}{800} \pi^4$$
Example

When $K = 4$, we have

$$a(1, 4) = -\frac{1}{720} + \frac{29}{4320} \pi^2 - \frac{427}{64800} \pi^4 + \frac{31}{18900} \pi^6$$

$$a(2, 4) = -\frac{16}{45} + \frac{208}{135} \pi^2 - \frac{2366}{2025} \pi^4 + \frac{496}{4725} \pi^6$$

$$a(3, 4) = -\frac{2187}{560} + \frac{2187}{160} \pi^2 - \frac{5103}{800} \pi^4 + \frac{2511}{4900} \pi^6$$

$$a(4, 4) = -\frac{2048}{315} + \frac{2048}{135} \pi^2 - \frac{12544}{2025} \pi^4 + \frac{15872}{33075} \pi^6$$

Other values $a(k, K)$ can be easily computed.
Conversion to Digital Filter

To convert the analog frequency response $A(f)$ into a digital frequency response $D(f)$, we can use the digital sinc function in place of the usual sinc function.

The digital sinc function $\text{dsinc}(f, N)$ can be written as

$$\text{dsinc}(f, N) := \frac{\sin(N\pi f)}{\sin(\pi f)}.$$ (4)

- The digital sinc function defined in (4) is periodic in $f$ with period 2:

$$\text{dsinc}(f + 2) = \text{dsinc}(f).$$

- We have the following approximation:

$$\text{sinc}(f) \approx \frac{1}{N} \text{dsinc} \left( \frac{f}{N}, N \right) \quad \text{for} \quad |f| < 0.5N.$$ for large values of $N$. 
Sinc vs. Digital Sinc

![Graphs of sinc(f) and dsinc(f/N)/N with N = 30](image)
Sinc Minus Digital Sinc

\[
sinc(f) - dsinc(f/N)/N \quad [N = 30]
\]
Sinc vs. Digital Sinc

\[ \text{sinc}(f) \approx \text{dsinc}(f/N)/N, \quad \text{for} \quad |f| < 0.5N \]

especially for large values of \( N \).

The design of digital differentiators described here is intended for long impulse responses. In this case, \( N \) is large and the approximation is valid.
Digital Lowpass Differentiators

Consider the function $D(f)$, based on the digital sinc function:

$$D(f) = \frac{1}{N} \sum_{k=1}^{K} a(k, K) \left[ \text{dsinc} \left( \frac{f - k}{N}, N \right) - \text{dsinc} \left( \frac{f + k}{N}, N \right) \right]$$

For $N > K$, we have the approximation

$$D(f) \approx A(f) \quad \text{for} \quad |f| < N/2.$$  

For example: with $K = 3$, $N = 30$: 

![Graph of A(f) - D(f)](image-url)
Digital Lowpass Differentiators

Consider the function $D(f)$, based on the digital sinc function:

$$D(f) = \frac{1}{N} \sum_{k=1}^{K} a(k, K) \left[ \text{sinc} \left( \frac{f - k}{N}, N \right) - \text{sinc} \left( \frac{f + k}{N}, N \right) \right]$$

- The function $D\left(\frac{N}{2\pi} \omega\right)$ is then a $2\pi$ periodic function of $\omega$ and can therefore be used as the frequency response $H(e^{j\omega})$ of a digital filter.
- Then $H(e^{j\omega})$ will be approximately maximally flat at $\omega = 0$. 
Digital Lowpass Differentiators

The impulse response \( h(n) \) is given by the inverse discrete-time Fourier transform of \( H(e^{j\omega}) \),

\[
h(n) = \text{IDTFT}\left\{e^{-j\left(\frac{N-1}{2}\right)\omega} \cdot D\left(\frac{N}{2\pi} \omega\right)\right\}
\]

where the phase term is included to make \( h(n) \) causal. Then \( h(n) \) is a linear-phase FIR impulse response of length \( N \):

\[
h(n) = \frac{1}{N^2} \sum_{k=1}^{K} a(k, K) \sin \left(\frac{2\pi k}{N} \left( n - \frac{(N-1)}{2}\right)\right)
\]

for \( 0 \leq n \leq N - 1 \). Once a table of values \( a(k, K) \) is computed, it can be used regardless of the length \( N \) of the impulse response \( h(n) \).

The width of the passband is controlled by the parameter \( K \), the number of flatness constraints at dc. Note that \( 2(K - 1) \) is the number of zeros of \( H(z) \) that lie away from the unit circle, as illustrated in the following examples.
Example $K = 2$
Example $K = 3$
Example $K = 4$
Example $K = 5$
Example $K = 6$
Example $K = 2$

Lowpass differentiator ($K = 2$, $N = 101$).
There are $2 = 2(K - 1)$ zeros contributing to the shape of the passband.
Example $K = 3$

Lowpass differentiator ($K = 3$, $N = 101$).
There are $4 = 2(K - 1)$ zeros contributing to the shape of the passband.
Example $K = 4$

Lowpass differentiator ($K = 4$, $N = 101$).
There are $6 = 2(K - 1)$ zeros contributing to the shape of the passband.
Example $K = 5$

Lowpass differentiator ($K = 5$, $N = 101$).

There are $8 = 2(K - 1)$ zeros contributing to the shape of the passband.
Lowpass differentiator \((K = 6, \ N = 101)\).

There are \(10 = 2(K - 1)\) zeros contributing to the shape of the passband.
Summary

1. Simple Design Algorithm

2. Efficient Implementation — Frequency Sampling

3. Still needed: Design rules — How to choose $N$ and $K$ so that specifications are satisfied.

4. Closed form formulas for $a(k, K)$?

5. Smaller stopband ripple can be achieved by appropriate modification.