USING THE DISCRETE FOURIER TRANSFORM

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DFT PROPERTIES

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{-kn}, \qquad 0 \le k \le N-1$$

$$x(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k)\, W_N^{kn}, \qquad 0 \le n \le N-1$$

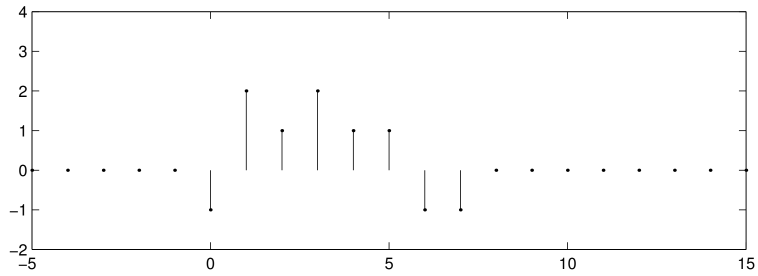
$$\boxed{W_N := e^{j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) + j\,\sin\left(\frac{2\pi}{N}\right)}$$

Periodicity	$X(k) = X(\langle k \rangle_N)$	$x(n) = x(\langle n \rangle_N)$
Circular Shift	$x(\langle n-m \rangle_N)$	$W_N^{-mk} \cdot X(k)$
Freq shift	$W_N^{mn} \cdot x(n)$	$X(\langle k-m \rangle_N)$
Circular conv	$x(n) \circledast g(n)$	$X(k) \cdot G(k)$
Modulation	$x(n) \cdot g(n)$	$\frac{1}{N} \cdot X(k) \circledast G(k)$
Time-reversal	$x(\langle -n \rangle_N)$	$X(\langle -k \rangle_N)$
Complex conj	$x^*(n)$	$X^*(\langle -k \rangle_N)$
Parseval's thm	$\sum_{n=0}^{N-1} x(n) \cdot g^*(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) \cdot G^*(k)$	

ZERO PADDING

Sec 3.3.3 in Mitra

Suppose $x(n)$ is supported on $0 \le n \le N-1$



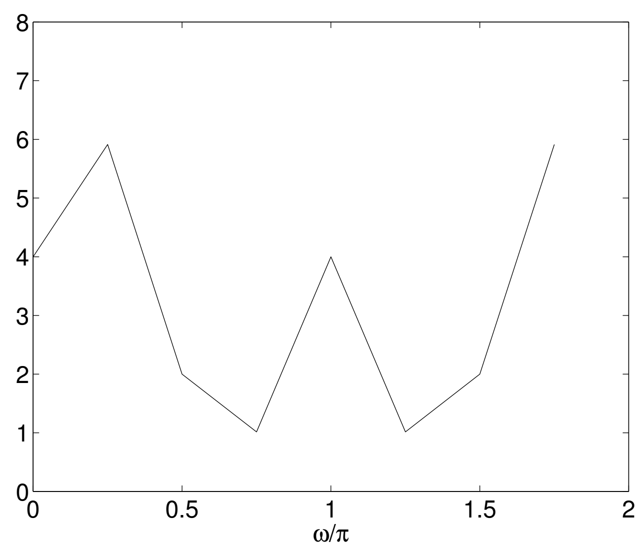
Problem: Make a plot of $X^f(\omega) = \text{DTFT}\{x(n)\}$.

Lets take the N-point DFT of $x(n)$,

$$X^f\left(\frac{2\pi}{N}k\right) = \text{DFT}\{[x(0), \ldots, x(N-1)]\}$$

In Matlab:

```
x = [-1 2 1 2 1 1 -1 -1];
N = length(x);
n = 0:N-1;
X = fft(x);
w = 2*pi*n/N;
plot(w/pi,abs(X));
```

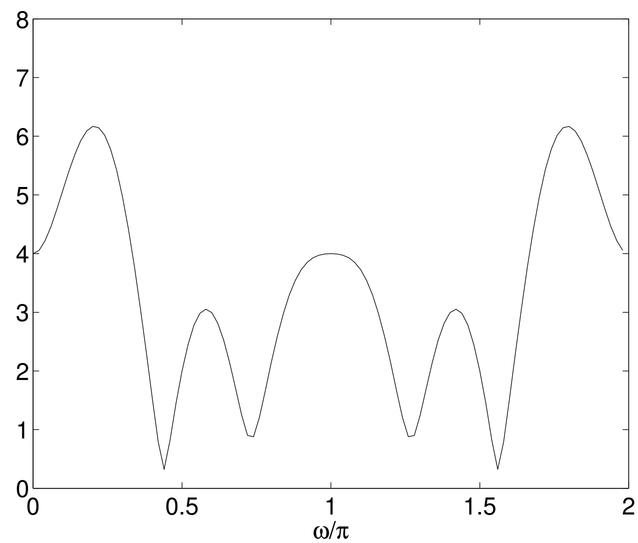


Better: Take an M-point DFT of $x(n)$, ($M >> N$)

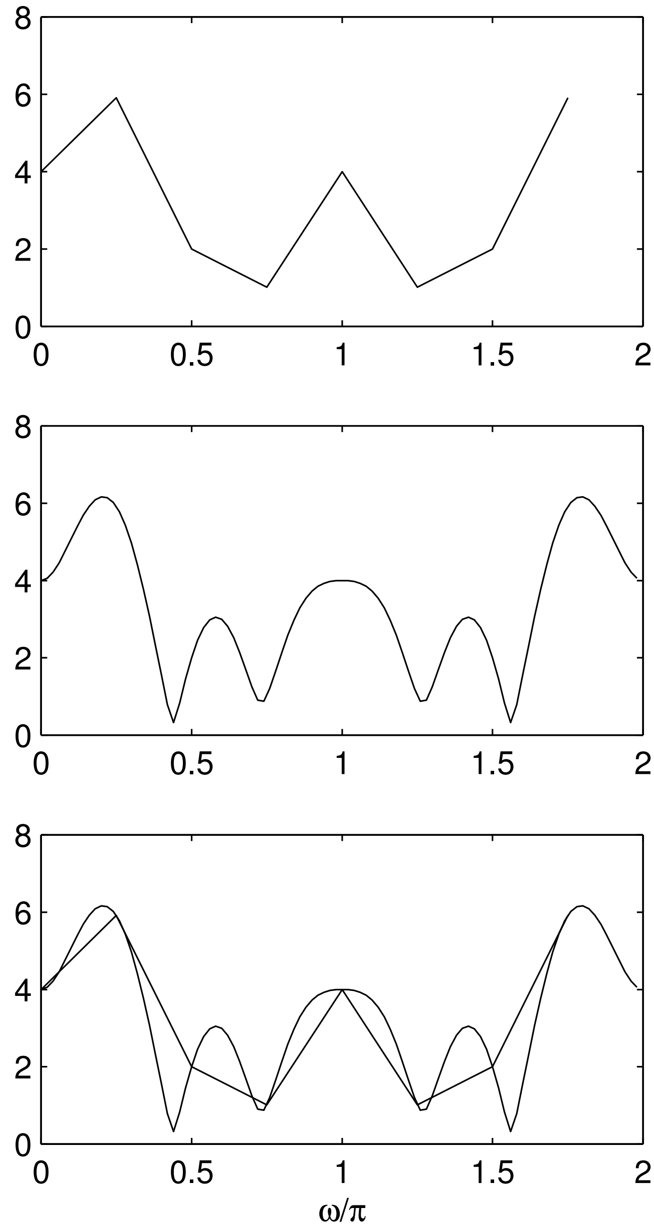
$$
\begin{aligned}
X^f\left(\frac{2\pi}{M}k\right) &= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{M}kn} \\
&= \sum_{n=0}^{N-1} x(n)\, W_M^{-kn} \\
&= \text{DFT}\{[x(0), \ldots, x(N-1), \underbrace{0, \ldots, 0}_{M-N}]\}
\end{aligned}
$$

That is: Zero-pad x and then take the DFT.
In Matlab:

```
x = [-1 2 1 2 1 1 -1 -1];
N = length(x);
M = 100;
m = 0:M-1;
X = fft([x zeros(1,M-N)]);
w = 2*pi*m/M;
plot(w/pi,abs(X));
```



Compare



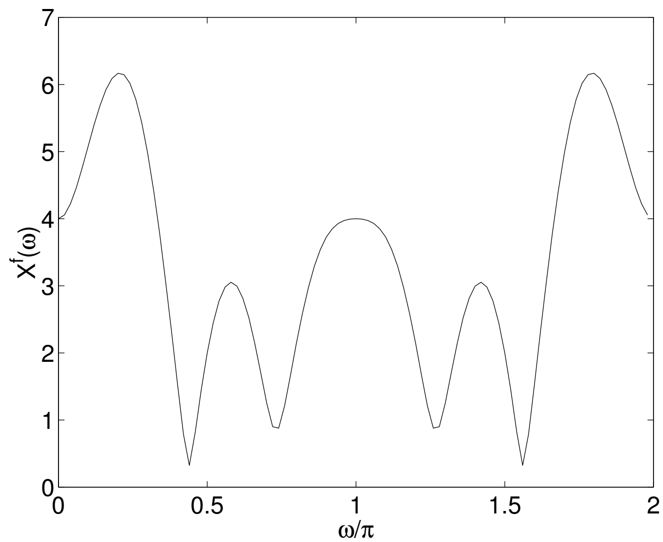
Zero-padding a sequence will result in a better plot of its DTFT. That is because the DFT samples the DTFT at the frequencies

$$\omega_k = \frac{2\pi}{L} k \qquad 0 \le k \le L-1$$

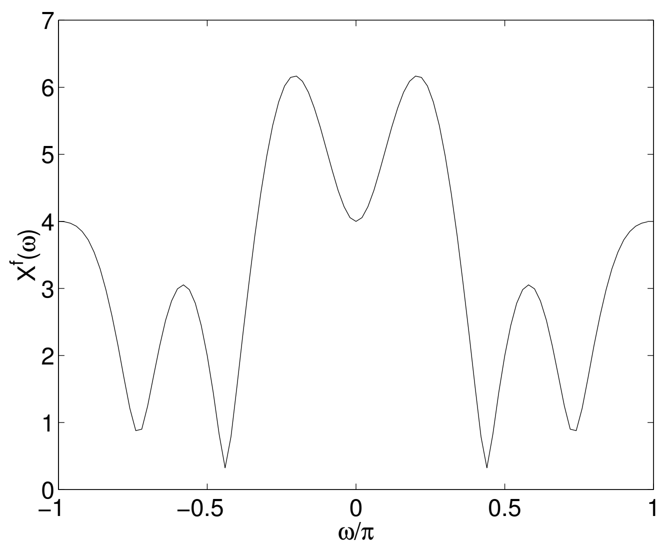
where L is the length of the DFT.

FFT SHIFT

The DFT samples the DTFT in the interval $0 \leq \omega \leq 2\pi$.



But it is more natural to plot the DTFT in the interval $-\pi \leq \omega \leq \pi$.



Then the DC component is in the middle of the spectrum.

In Matlab, the command `fftshift` can be use for this.

In Matlab, the command `fftshift(X)` swaps the left and right halves of X.

```
x = [-1 2 1 2 1 1 -1 -1];
N = length(x);
M = 100;
m = 0:M-1;
X = fft([x zeros(1,M-N)]);
w = 2*pi*m/M;
X = fftshift(X);             % shift spectrum around
w = w-pi;                    % modify w accordingly
plot(w/pi,abs(X));
xlabel('\omega/\pi')
ylabel('|X^f(\omega)|')
```

`fftshift` is useful for visualizing the Fourier transform with the DC component in the middle of the spectrum.

Note: In Matlab,

```
X = fft([x, zeros(1,M-N)]);
```

can be abbreviated as

```
X = fft(x,M);
```

PHYSICAL FREQUENCY

The notation $x(n)$ hides the physical sampling frequency.

Suppose an analog signal $x_a(t)$ is sampled at F_s Hz,

$$x(n) = x_a(n\,T_s),$$

with

$$F_s = \frac{1}{T_s}.$$

If N samples are collected then we have a finite-length discrete-time signal $x(n)$, $0 \le n \le N-1$.

We may take the DFT of this N-point signal,

$$X^d(k) = \text{DFT}\left\{x(n)\right\}.$$

Question:

What is the physical frequency of the DFT coefficient $X^d(k)$?

Recall that when an analog signal $x_a(t)$ is sampled,

$$x_s(t) = \sum_n x_a(n\,T_s)\,\delta(t - n\,T_s)$$

the new frequency spectrum $X_s(\omega)$ is periodic with period ω_s Rad, or F_s Hz. ($\omega_s = 2\,\pi\,F_s$).

Similarly, $X^f(\omega) = \text{DTFT}\,\{x(n)\}$ is periodic in ω with period 2π.

So the conversion relation between the DTFT of $x(n)$ and the spectrum of the original analog signal $x_a(t)$ is

$$2\pi\,\text{Rad} = F_s\,\text{Hz}.$$

That is

$$\frac{F_s\,\text{Hz}}{2\pi\,\text{Rad}} = 1.$$

(Usually the Rad units are not explicitly stated, but here it is convenient to do so.)

Therefore, the DFT coefficient $X^d(k)$ corresponds to frequency

$$\frac{2\pi}{N}\,k\,\text{Rad} = \frac{2\pi}{N}\,k\,\text{Rad}\cdot\frac{F_s\,\text{Hz}}{2\pi\,\text{Rad}} = \frac{F_s}{N}\,k\,\text{Hz}$$

The physical frequency of the DFT coefficient $X^d(k)$ is $\dfrac{F_s}{N}\,k$ Hz where F_s is the sampling frequency in Hz.

Example

Suppose the analog signal $x_a(t)$ is sampled at 4 Hz for 2 seconds, resulting in the 8 samples,

$$x(n) = x_a(n\,T_s) = [-1,\ 2,\ 1,\ 2,\ 1,\ 1,\ -1,\ -1]$$

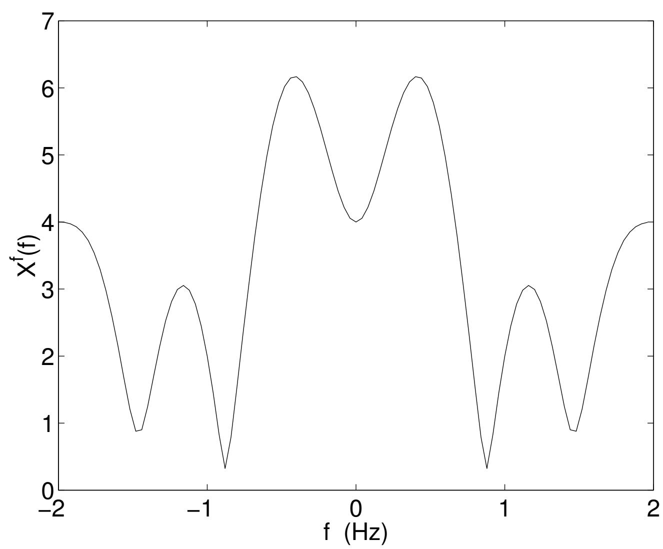
for $0 \le n \le 7$.

Problem

Using Matlab, plot the spectrum $X^f(\omega) = \text{DTFT}\,\{x(n)\}$ versus physical frequency in Hz with the DC component in the center.

Solution

```
Fs = 4;
x = [-1 2 1 2 1 1 -1 -1];
M = 100;
m = 0:M-1;
X = fftshift(fft(x,M));
f = Fs*m/M-Fs/2;
plot(f,abs(X));
xlabel('f (Hz)')
ylabel('X^f(f)')
```



RESOLUTION OF THE DFT

The *frequency resolution* of the DFT is the spacing between two adjacent frequencies: $\Delta\omega = 2\pi/N$.

This is also called the *frequency bin* of the DFT.

The corresponding physical frequency resolution, in Hertz, is

$$\Delta f = \frac{F_s}{N} = \frac{1}{N\,T_s}.$$

Note: $N\,T_s$ is the total duration of the original continuous-time signal. Therefore,

> the physical frequency resolution (in Hz) of the DFT is the inverse of the signal duration (in Sec).

Sec 8.3.3 in Mitra

Suppose a sinusoidal signal of unknown frequency f_o Hz is sampled at F_s Hz and N samples are collected.

An important problem in DSP is to determine the unknown frequency f_o from the samples $x(n)$. (For example, in DTMF.)

Lets use the DFT.

Example 1

A 10 Hz sinusoid is sampled at 64 Hz (no aliasing occurs) for 0.5 seconds. Therefore 32 samples are collected.

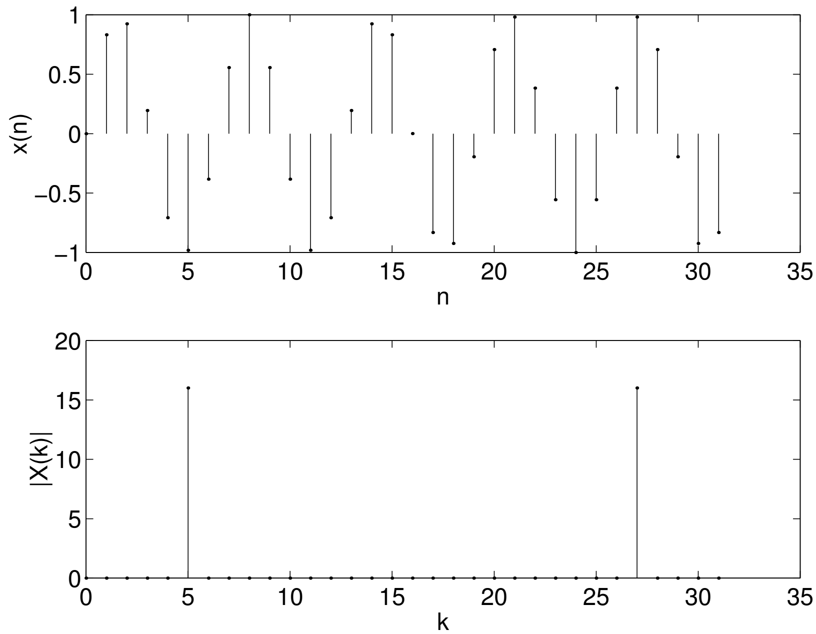
$$x(n) = \cos(2\pi f_o n T_s), \qquad 0 \le n \le N-1$$

where $f_o = 10$, $F_s = 64$, $T_s = 1/F_s$, and $N = 32$.

Let us examine the DFT $X^d(k) = \text{DFT}\,\{x(n)\}$.

The following Matlab code computes the DFT of the discrete-time signal and makes a stem plot of the DFT.

```
fo = 10;
Fs = 64;
Ts = 1/Fs;
N = 32;
n = 0:N-1;
x = cos(2*pi*fo*n*Ts);
X = fft(x);
subplot(2,1,1)
stem(n,x,'.')
xlabel('n')
ylabel('x(n)')
subplot(2,1,2)
stem(n,abs(X),'.')
xlabel('k')
ylabel('|X(k)|')
```



Suppose we had not known the frequency f_o of the sinusoid; how can we find it from the DFT values?

It can be seen that $|X^d(k)|$ has a peak at DFT index $k = 5$ and $k = 27$. The physical frequency corresponding to $k = 5$ is

$$\frac{F_s}{N}\,k = \frac{64}{32}\,5 = 10\,\text{Hz}.$$

That agrees with the true value.
The physical frequency corresponding to $k = 27$ is

$$\frac{F_s}{N}\,k = \frac{64}{32}\,27 = 54\,\text{Hz}.$$

Recalling that the spectrum of the sampled signal is periodic with period $F_s = 64$ Hz, this frequency corresponds to $54 - 64$ Hz $=$ -10 Hz. That frequency corresponds to the negative side of the spectrum. Therefore, it also agrees with the true frequency.

In this example, the unknown frequency f_o can be read from the DFT.

Example 2

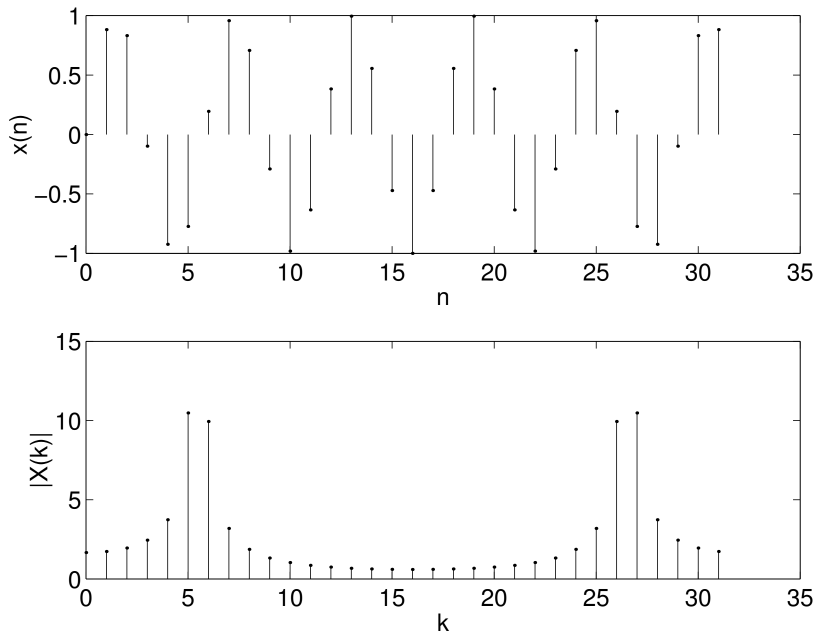
An 11 Hz sinusoid is sampled at 64 Hz (no aliasing occurs) for 0.5 seconds. Therefore 32 samples are collected.

$$x(n) = \cos(2\pi f_o n T_s), \qquad 0 \le n \le N-1$$

where $f_o = 11$, $F_s = 64$, $T_s = 1/F_s$, and $N = 32$.

(We change only f_o.)

Let us examine the DFT $X^d(k) = \text{DFT}\{x(n)\}$.



We might expect that the DFT is zero except for the 'right' values of k as before.

k would be found by:

$$\frac{F_s}{N} k = 11 \quad \text{or} \quad k = \frac{11 \cdot N}{F_s} = \frac{11 \cdot 32}{64} = 5.5$$

which is not an integer. Therefore, the largest values of $X^d(k)$ occur at $k = 5$ and $k = 6$. (And at $k = 26$ and $k = 27$ representing the negative frequencies.)

The frequency 'leaks' into other DFT bins.

Sec 8.3.3 in Mitra

Suppose we have an analog signal,

$$x_a(t) = \cos(2\pi f_o t)$$

which is sampled at F_s Hz,

$$x_1(n) = x_a(n\,T_s)$$

where

$$F_s = \frac{1}{T_s}.$$

In practice, we collect only a finite number of samples,

$$x_2(n) = \begin{cases} x_a(n\,T_s) & 0 \le n \le N-1 \\ 0 & \text{otherwise.} \end{cases}$$

It is important to note that the spectrums of $x_1(n)$ and $x_2(n)$ are different.

$$X_1^f(\omega) = \text{DTFT}\,\{x_1(n)\} \neq X_2^f(\omega) = \text{DTFT}\,\{x_2(n)\}$$

Suppose we take the DFT of the collected samples,

$$\begin{aligned} X^d(k) &= \text{DFT}\,\{[x_1(0), \dots, x_1(N-1)]\} \\ &= \text{DFT}\,\{[x_2(0), \dots, x_2(N-1)]\}\,. \end{aligned}$$

What does $X^d(k)$ represent?

From the definitions of the DFT and DTFT,

$$X^d(k) = X_2^f\left(\frac{2\pi}{N}\,k\right) \neq X_1^f\left(\frac{2\pi}{N}\,k\right)$$

The DFT values $X^d(k)$ are samples of the spectrum of $x_2(n)$, not of the spectrum of $x_1(n)$.

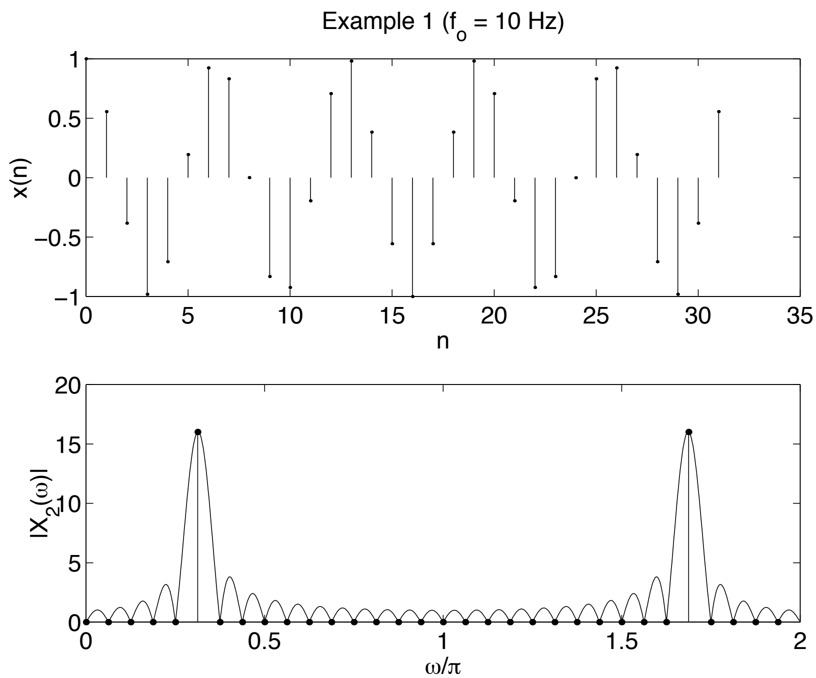
What does $X_2^f(\omega)$ look like?

How can we make a plot of $X_2^f(\omega)$?

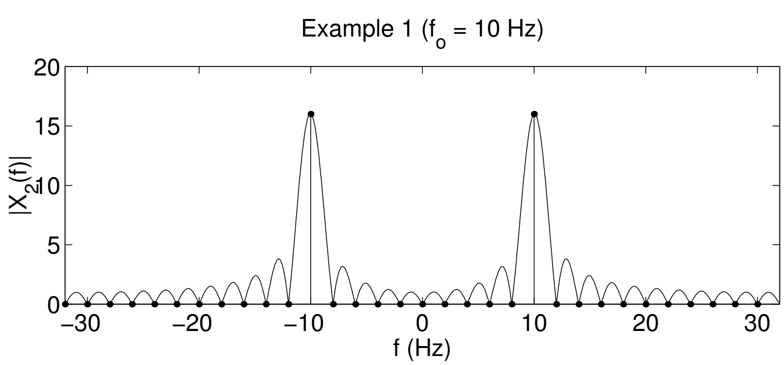
As before: Evaluate it on a dense set of frequencies $\omega_k = \frac{2\pi}{M}k$, $0 \le k \le M-1$ where $M >> N$.

$\Longrightarrow$ zero-pad the samples and take the DFT:

$$X_2^f\left(\frac{2\pi}{M}k\right) = \mathsf{DFT}\{[x(0), \dots, x(N-1), \underbrace{0, \dots, 0}_{M-N}]\}$$

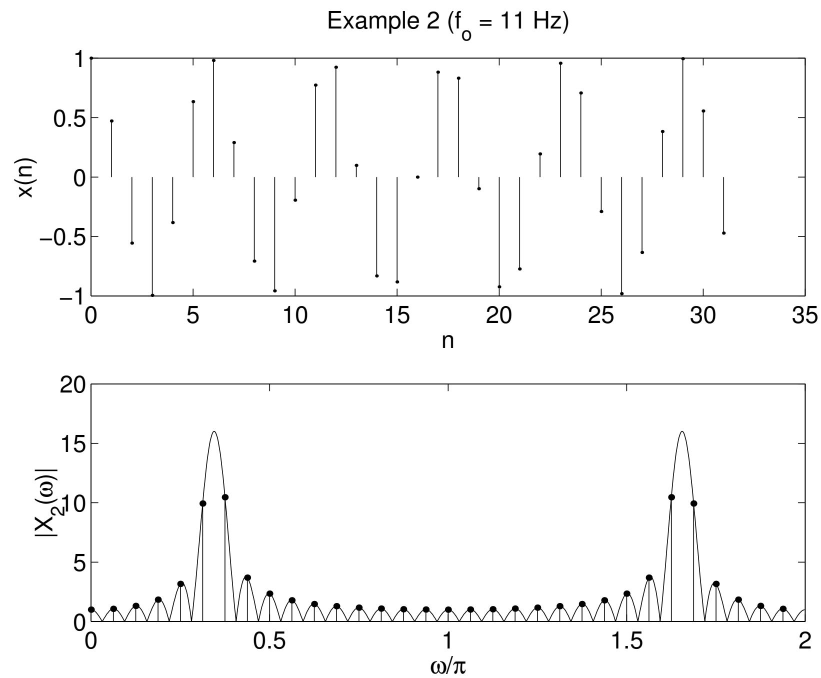


Lets use physical frequency with the DC component in the center.

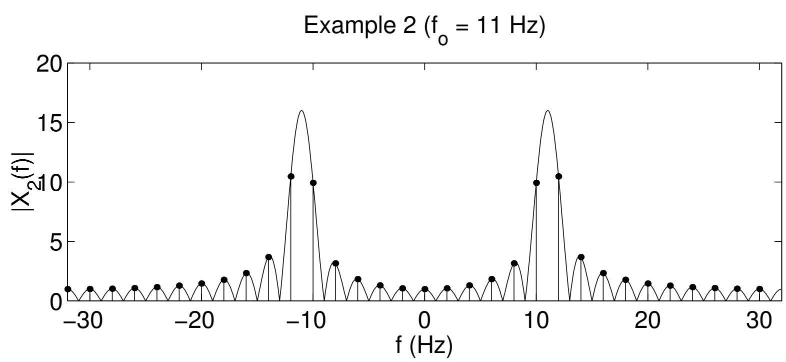


```
fo = 10;
Fs = 64;
Ts = 1/Fs;
N = 32;
n = 0:N-1;
x = cos(2*pi*fo*n*Ts);   % create N samples

X = fft(x);              % DFT of 32 samples
w = 2*pi*n/N;            % frequency axis

M = 2^10;                % choose M >> N
m = 0:M-1;
XM = fft(x,M);           % zero-pad and calculate the DFT
wM = 2*pi*m/M;           % frequency axis

stem(w/pi,abs(X),'.')    % make plots
hold on
plot(wM/pi,abs(XM))
hold off
xlabel('\omega/\pi')
ylabel('|X(\omega)|')
```

```
% PUT THE DC COMPONENT IN CENTER
% USE PHYSICAL FREQUENCY

fo = 10;
Fs = 64;
Ts = 1/Fs;
N = 32;
n = 0:N-1;
x = cos(2*pi*fo*n*Ts);   % create N samples

X = fftshift(fft(x));    % DFT of 32 samples
f = Fs*n/N-Fs/2;         % frequency axis

M = 2^10;                % choose M >> N
m = 0:M-1;
XM = fftshift(fft(x,M)); % zero-pad and calculate the DFT
fM = Fs*m/M-Fs/2;        % frequency axis

stem(f,abs(X),'.')       % make plots
hold on
plot(fM,abs(XM))
hold off
xlabel('f (Hz)')
ylabel('|X(f)|')
```



Lets use physical frequency with the DC component in the center.



In Example 1, the DFT samples $X^d(k)$ coincide with the nulls of $X_2^f(\omega)$ and with its maximum value. In Example 2, they do not.

For Example 2, we can still obtain the frequency f_o, by locating the maximum of $|X_2^f(\omega)|$.

The DTFT of $x_1(n)$ is:

$$X_1^f(\omega) = \pi\,\delta(\omega - \omega_o) + \pi\,\delta(\omega + \omega_o), \quad \text{for } |\omega| \le \pi$$

where $\omega_o = 2\pi f_o T_s$.

To get the DTFT of $x_2(n)$, note that

$$x_2(n) = x_1(n) \cdot s(n)$$

where

$$s(n) := \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise.} \end{cases}$$

Using the modulation property of the DTFT,

$$\boxed{X_2^f(\omega) = \frac{1}{2\pi}\, X_1^f(\omega) \circledast S^f(\omega)}$$

where

$$X_1^f(\omega) \circledast S^f(\omega) := \int_{-\pi}^{\pi} X_1^f(\theta)\; S^f(\omega - \theta)\, d\theta$$

and

$$S^f(\omega) = \text{DTFT}\left\{s(n)\right\}.$$

Therefore,

$$\begin{aligned} X_2^f(\omega) &= \frac{1}{2\pi}\left(\pi\,\delta(\omega - \omega_o) + \pi\,\delta(\omega + \omega_o)\right) \circledast S^f(\omega) \\ &= \frac{1}{2}\, S^f(\omega - \omega_o) + \frac{1}{2}\, S^f(\omega + \omega_o). \end{aligned}$$

$$X_1^f(\omega) = \pi\,\delta(\omega - \omega_o) + \pi\,\delta(\omega + \omega_o), \quad \text{for } |\omega| \le \pi$$

$$X_2^f(\omega) = \frac{1}{2}\,S^f(\omega - \omega_o) + \frac{1}{2}\,S^f(\omega + \omega_o)$$

DIGITAL SINC FUNCTION

The rectangle function is given by

$$s(n) := \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise.} \end{cases}$$

The DTFT of $s(n)$ is $S_1^f(\omega)$ is given by

$$\begin{aligned} S_1^f(\omega) &= \text{DTFT}\{s(n)\} \\ &= \sum_{n=-\infty}^{\infty} s(n)\, e^{-jn\omega} \\ &= \sum_{n=0}^{N-1} e^{-jn\omega} \\ &= \frac{1 - e^{-jN\omega}}{1 - e^{-j\omega}} \qquad \text{(using the geometric sum formula)} \\ &= \frac{e^{-j\frac{N}{2}\omega}\left(e^{j\frac{N}{2}\omega} - e^{-j\frac{N}{2}\omega}\right)}{e^{-j\frac{1}{2}\omega}\left(e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega}\right)} \\ &= e^{-j\frac{N-1}{2}\omega} \cdot \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} \end{aligned}$$

$$\frac{\sin\left(\frac{N}{2}\,\omega\right)}{\sin\left(\frac{1}{2}\,\omega\right)} \text{ is called the Digital Sinc Function.}$$