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## DFT DEFINITION

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The DFT of an  $N$ -point signal

$$x[n], \quad n \in \mathbb{Z}_N := \{0, 1, \dots, N-1\}$$

is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{-kn}, \quad k \in \mathbb{Z}_N \quad (1)$$

where

$$W_N = e^{j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) + j \sin\left(\frac{2\pi}{N}\right)$$

is the principal  $N$ -th root of unity.

The original sequence  $x[n]$  can be retrieved by the inverse discrete Fourier transform (IDFT)

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{kn}, \quad n \in \mathbb{Z}_N.$$

## THE MOD NOTATION

---

The notation  $\langle k \rangle_N$  denotes the remainder  $r$  when  $k$  is divided by  $N$ . This is also denoted as  $k \bmod N$ . For example

$$\langle 3 \rangle_4 = 3 \quad \text{and} \quad \langle 6 \rangle_4 = 2.$$

The following table shows  $\langle k \rangle_4$  for values of  $k$  from 0 to 7.

$k$	$\langle k \rangle_4$
0	0
1	1
2	2
3	3
4	0
5	1
6	2
7	3
$\vdots$	$\vdots$

The notation  $\langle k \rangle_N$  is also defined for negative integers  $k$ . We choose the remainder  $r$  so that it is between 0 and  $N - 1$ .

$$0 \leq \langle k \rangle_N \leq N - 1$$

For example,

$$-3 = (-1)(4) + 1 \quad \text{so} \quad \langle -3 \rangle_4 = 1$$

and

$$-6 = (-2)(4) + 2 \quad \text{so} \quad \langle -6 \rangle_4 = 2$$

## THE MOD NOTATION

---

The following table shows  $\langle k \rangle_4$  for values of  $k$  from -8 to 7.

$k$	$\langle k \rangle_4$
$\vdots$	$\vdots$
-8	0
-7	1
-6	2
-5	3
-4	0
-3	1
-2	2
-1	3
0	0
1	1
2	2
3	3
4	0
5	1
6	2
7	3
$\vdots$	$\vdots$

In MATLAB, we can generate  $\langle k \rangle_N$  using the command `mod`, as illustrated by the code fragment:

## THE MOD NOTATION

---

```
>> k=-8:7;  
>> [k', mod(k,4)']
```

```
ans =
```

```
-8    0  
-7    1  
-6    2  
-5    3  
-4    0  
-3    1  
-2    2  
-1    3  
 0    0  
 1    1  
 2    2  
 3    3  
 4    0  
 5    1  
 6    2  
 7    3
```

Evidently  $\langle k \rangle_N$  is a periodic function of  $k$  with period  $N$ :

$$\langle k + N \rangle_N = \langle k \rangle_N$$

## PERIODICITY OF $W_N$

---

Our definition of  $W_N$  is

$$W_N := e^{j\frac{2\pi}{N}}.$$

If we write  $k$  as  $k = l \cdot N + r$  with  $r \in \mathbb{Z}_N$  then we have  $\langle k \rangle_N = r$  and

$$\begin{aligned} W_N^k &= e^{j\frac{2\pi}{N}(l \cdot N + r)} \\ &= e^{j\frac{2\pi}{N}(l \cdot N)} \cdot e^{j\frac{2\pi}{N}r} \\ &= 1 \cdot W_N^r \\ &= W_N^{\langle k \rangle_N}. \end{aligned}$$

Therefore  $W_N^k$  is a periodic function of  $k$  with a period of  $N$ . The identity

$$\boxed{W_N^k = W_N^{\langle k \rangle_N}}$$

is useful when working with the DFT.

A related identity is

$$\boxed{W_N^{m \cdot k} = W_N^{m \cdot \langle k \rangle_N}}$$

which is derived as

$$\begin{aligned} W_N^{m \cdot k} &= e^{j\frac{2\pi}{N}(m)(l \cdot N + r)} \\ &= e^{j\frac{2\pi}{N}(m \cdot l \cdot N)} \cdot e^{j\frac{2\pi}{N}m \cdot r} \\ &= 1 \cdot W_N^{m \cdot r} \\ &= W_N^{m \cdot \langle k \rangle_N}. \end{aligned}$$

## A USEFUL IDENTITY

---

First, recall the geometric summation formula:

$$\sum_{n=0}^{N-1} a^n = \begin{cases} \frac{1-a^N}{1-a} & a \neq 1 \\ N & a = 1. \end{cases}$$

If  $a = W_N^k$ , then we get:

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} \frac{1-W_N^{Nk}}{1-W_N^k} & W_N^k \neq 1 \\ N & W_N^k = 1. \end{cases}$$

Because  $W_N^{Nk} = 1$ , this simplifies to

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} 0 & W_N^k \neq 1 \\ N & W_N^k = 1. \end{cases}$$

Note that  $W_N^k = 1$  if  $k = 0, \pm N, \pm 2N, \dots$ . In other words  $W_N^k = 1$  if  $\langle k \rangle_N = 0$ . Similarly  $W_N^k \neq 1$  if  $\langle k \rangle_N \neq 0$ . Therefore we can simplify the summation further to obtain:

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} 0 & \langle k \rangle_N \neq 0 \\ N & \langle k \rangle_N = 0. \end{cases}$$

Using the discrete-time delta function  $\delta[n]$ , defined as

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0, \end{cases}$$

we finally get the sought identity in the simple form:

$$\boxed{\sum_{n=0}^{N-1} W_N^{nk} = N \cdot \delta[\langle k \rangle_N].}$$

## INVERSE DFT PROOF

---

With the formula

$$\sum_{n=0}^{N-1} W_N^{nk} = N \cdot \delta[\langle k \rangle_N]$$

we are ready to verify the the inverse DFT formula.

To verify the inversion formula, we substitute the DFT into the inverse DFT formula:

$$\begin{aligned} x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{kn} \\ x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} \left( \sum_{l=0}^{N-1} x[l] W_N^{-kl} \right) W_N^{kn}, \\ &= \frac{1}{N} \sum_{l=0}^{N-1} x[l] \cdot \sum_{k=0}^{N-1} W_N^{k(n-l)}, \\ &= \frac{1}{N} \sum_{l=0}^{N-1} x[l] \cdot N \cdot \delta[\langle n-l \rangle_N], \\ &= x[\langle n \rangle_N] \\ &= x[n] \quad \text{provided } 0 \leq n \leq N-1. \end{aligned}$$



## CIRCULAR SHIFTING

---

Given an  $N$ -point signal  $\{x[n], n \in \mathbb{Z}_N\}$ , the signal

$$g[n] := x[\langle n - m \rangle_N]$$

represents a circular shift of  $x[n]$  by  $m$  samples to the right. For example, if

$$g[n] := x[\langle n - 1 \rangle_N]$$

then

$$g[0] = x[\langle -1 \rangle_N] = x[N - 1]$$

$$g[1] = x[\langle 0 \rangle_N] = x[0]$$

$$g[2] = x[\langle 1 \rangle_N] = x[1]$$

$\vdots$

$$g[N - 1] = x[\langle N - 2 \rangle_N] = x[N - 2]$$

For example, if  $x[n]$  is the 4-point signal

$$x[n] = (1, 3, 5, 2)$$

then

$$x[\langle n - 1 \rangle_N] = (2, 1, 3, 5).$$

$x[\langle n - m \rangle_N]$  represents a *circular* shift by  $m$  samples.

## CIRCULAR SHIFTING

---

The following MATLAB code fragment illustrates the circular shift.

```
>> g = [1 3 5 2]
g =
     1     3     5     2

>> n = 0:3
n =
     0     1     2     3

>> mod(n-1,4)
     3     0     1     2

>> g(mod(n-1,4)+1)    % Add 1 to account for MATLAB indexing
     2     1     3     5
```

Because indexing in MATLAB begins with 1 rather than with 0, it is necessary to add 1 to the index vector.

## CIRCULAR CONVOLUTION

---

The usual convolution (or linear convolution) is defined as

$$x[n] * g[n] := \sum_{m=0}^{N-1} x[m]g[n-m].$$

But when  $x[n]$  and  $g[n]$  are defined as  $N$ -point signals

$$x[n], \quad n \in \mathbb{Z}_N \quad \text{and} \quad g[n], \quad n \in \mathbb{Z}_N$$

then the term  $g[n-m]$  will fall outside the range. The definition of *circular* convolution evaluates the index modulo  $N$ :

$$x[n] \circledast g[n] := \sum_{m=0}^{N-1} x[m]g[\langle n-m \rangle_N].$$

The following MATLAB function illustrates how the circular convolution can be computed.

```
function y = cconv(x,g)
% y = cconv(x,g)

N = length(x);
n = 0:N-1;
y = zeros(size(x));
for m = 0:N-1
    y = y + x(m+1)*g(mod(n-m,N)+1);
end
```

For example:

```
>> x = [2 4 3 1 5];
>> g = [7 3 5 2 4];
>> y = cconv(x,g)
y =
    56    73    57    60    69
```

## CIRCULAR TIME-REVERSAL

---

Given an  $N$ -point signal  $\{x[n], n \in \mathbb{Z}_N\}$ , the signal

$$g[n] := x[\langle -n \rangle_N]$$

is given by

$$g[n] = \begin{cases} x[0] & n = 0 \\ x[N - n] & 1 \leq n \leq N - 1. \end{cases}$$

as shown by:

$$g[0] = x[\langle 0 \rangle_N] = x[0]$$

$$g[1] = x[\langle -1 \rangle_N] = x[N - 1]$$

$$g[2] = x[\langle -2 \rangle_N] = x[N - 2]$$

⋮

$$g[N - 1] = x[\langle 1 - N \rangle_N] = x[1].$$

For example, if  $x[n]$  is the 4-point signal

$$x[n] = (1, 3, 5, 2)$$

then

$$x[\langle -n \rangle_N] = (1, 2, 5, 3).$$

## CIRCULAR TIME-REVERSAL

---

The following MATLAB code fragment illustrates circular time-reversal for a 4-point signal.

```
>> g = [1 3 5 2]
```

```
g =
```

```
    1    3    5    2
```

```
>> N = 4;
```

```
>> n = 0:N-1;
```

```
>> g(mod(-n,N)+1)
```

```
ans =
```

```
    1    2    5    3
```

## CIRCULAR SYMMETRY

---

A discrete-time signal  $x[n]$  is symmetric if

$$x[n] = x[-n].$$

But if  $x[n]$  is an  $N$ -point signal  $\{x[n], n \in \mathbb{Z}_N\}$ , then the sample  $x[-n]$  falls outside the range. The definition of *circular* symmetry evaluates the index modulo  $N$ . An  $N$ -point signal  $x[n]$  is circularly symmetric if

$$x[n] = x[\langle -n \rangle_N].$$

For example, the sequences

$$x[n] = (5, 3, 4, 2, 2, 4, 3)$$

$$v[n] = (5, 3, 4, 2, 7, 2, 4, 3)$$

are both circularly symmetric.

## PERIODICITY PROPERTY OF THE DFT

---

Given the  $N$ -point signal  $\{x[n], n \in \mathbb{Z}_N\}$ , we defined the DFT coefficients  $X[k]$  for  $0 \leq k \leq N - 1$ . But if  $k$  lies *outside* the range  $0, \dots, N - 1$ , then

$$\boxed{X[k] = X[\langle k \rangle_N].}$$

To derive this equation, write  $k$  as  $k = l \cdot N + r$  with  $r \in \mathbb{Z}_N$ . Then  $\langle k \rangle_N = r$ , and

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{-n \cdot (l \cdot N + r)} \\ &= \sum_{n=0}^{N-1} x[n] W_N^{-n l N} \cdot W_N^{-nr} \\ &= \sum_{n=0}^{N-1} x[n] W_N^{-nr} \\ &= X[r] \\ &= X[\langle k \rangle_N]. \end{aligned}$$

If the DFT formula is evaluated for  $k$  outside the range  $k \in \mathbb{Z}_N$ , then one finds that  $X[k]$  is periodic with period  $N$ .

## PERIODICITY PROPERTY OF THE DFT

---

Likewise, given the  $N$ -point DFT vector  $\{X[k], k \in \mathbb{Z}_N\}$ , we defined the *Inverse* DFT samples  $x[n]$  for  $n \in \mathbb{Z}_N$ . But if  $n$  lies *outside* the range  $0, \dots, N - 1$ , then

$$\boxed{x[n] = x[\langle n \rangle_N].}$$

To derive this equation, write  $n$  as  $n = l \cdot N + r$  with  $r \in \mathbb{Z}_N$ . Then  $\langle n \rangle_N = r$ , and

$$\begin{aligned} x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{k \cdot (l \cdot N + r)} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{k l N} \cdot W_N^{kr} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{kr} \\ &= x[r] \\ &= x[\langle n \rangle_N]. \end{aligned}$$

If the Inverse DFT formula is evaluated for  $n$  outside the range  $n \in \mathbb{Z}_N$ , then one finds that  $x[n]$  is periodic with period  $N$ .



## CIRCULAR SHIFT PROPERTY OF THE DFT

---

If

$$G[k] := W_N^{-mk} \cdot X[k]$$

then

$$g[n] = x[\langle n - m \rangle_N].$$

**Derivation:**

Begin with the Inverse DFT.

$$\begin{aligned} g[n] &= \frac{1}{N} \sum_{k=0}^{N-1} G[k] W_N^{nk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-mk} X[k] W_N^{nk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{k(n-m)} \\ &= x[n - m] \\ &= x[\langle n - m \rangle_N]. \end{aligned}$$

## CIRCULAR SHIFT PROPERTY OF THE DFT

---

The following MATLAB code fragment illustrates the circular shift property with a shift of 2 samples. property.

```
>> x = [3 1 5 2 4]';  
>>  
>> N = length(x); % N: data length  
>> W = exp(2*pi/N*sqrt(-1));  
>> X = fft(x);  
>> k = [0:N-1]'; % frequency index  
>> m = 2; % shift  
>> G = W.^(-m*k) .* X;  
>> g = ifft(G)
```

g =

```
2.0000 - 0.0000i  
4.0000 + 0.0000i  
3.0000 - 0.0000i  
1.0000 - 0.0000i  
5.0000 - 0.0000i
```

## MODULATION PROPERTY (circular shift in frequency)

---

If

$$g[n] := W_N^{mn} \cdot x[n]$$

then

$$G[k] = X[\langle k - m \rangle_N].$$

### Derivation:

Begin with the DFT.

$$\begin{aligned} G[k] &= \sum_{n=0}^{N-1} g[n] W_N^{-nk} \\ &= \sum_{n=0}^{N-1} W_N^{mn} x[n] W_N^{-nk} \\ &= \sum_{n=0}^{N-1} x[n] W_N^{-n(k-m)} \\ &= X[k - m] \\ &= X[\langle k - m \rangle_N]. \end{aligned}$$

## CIRCULAR CONVOLUTION PROPERTY OF THE DFT

---

If

$$y[n] := \sum_{m=0}^{N-1} x[m]g[\langle n - m \rangle_N]$$

then

$$Y[k] = X[k] \cdot G[k].$$

**Derivation:**

$$\begin{aligned} Y[k] &= \sum_{n=0}^{N-1} y[n]W_N^{-nk} \\ &= \sum_{n=0}^{N-1} \cdot \sum_{m=0}^{N-1} x[m]g[\langle n - m \rangle_N]W_N^{-nk} \\ &= \sum_{m=0}^{N-1} x[m] \cdot \sum_{n=0}^{N-1} g[\langle n - m \rangle_N]W_N^{-nk} \\ &= \sum_{m=0}^{N-1} x[m]W_N^{-mk}G[k] \\ &= G[k] \cdot \sum_{m=0}^{N-1} x[m]W_N^{-mk} \\ &= G[k] \cdot X[k] \end{aligned}$$

## CIRCULAR CONVOLUTION PROPERTY OF THE DFT

---

The following MATLAB code fragment illustrates the DFT convolution theorem.

```
>> x = [2 4 3 1 5];
>> g = [7 3 5 2 4];
>> cconv(x,g)

ans = [ 56 73 57 60 69 ]

>> ifft(fft(x).*fft(g))'

ans =

56.0000 + 0.0000i
73.0000 - 0.0000i
57.0000 + 0.0000i
60.0000 - 0.0000i
69.0000 - 0.0000i
```

## PARSEVAL'S THEOREM FOR THE DFT

---

$$\sum_{n=0}^{N-1} x[n] \cdot g^*[n] = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X[k] \cdot G^*[k]$$

**Derivation:**

$$\begin{aligned} \sum_{n=0}^{N-1} x[n] \cdot g^*[n] &= \sum_{n=0}^{N-1} x[n] \cdot \left( \frac{1}{N} \sum_{k=0}^{N-1} G[k] W_N^{nk} \right)^* \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot \sum_{k=0}^{N-1} G^*[k] W_N^{-nk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} G^*[k] \cdot \sum_{n=0}^{N-1} x[n] W_N^{-nk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot G^*[k] \end{aligned}$$

## PARSEVAL'S THEOREM FOR THE DFT

---

The following MATLAB code fragment illustrates Parseval's theorem.

```
>> N = 4;
>> x = randn(1,N) + i*randn(1,N)

x =
    0.7812 - 1.1878i    0.5690 - 2.2023i   -0.8217 + 0.9863i   -0.2656 - 0.5186i

>> g = randn(1,N) + i*randn(1,N)

g =
    0.3274 - 0.9471i    0.2341 - 0.3744i    0.0215 - 1.1859i   -1.0039 - 1.0559i

>> X = fft(x);
>> G = fft(g);
>> sum(x.*conj(g))

ans =
    1.9655 - 0.6644i

>> sum(X.*conj(G))/N

ans =
    1.9655 - 0.6644i
```

## TIME-REVERSAL PROPERTY OF THE DFT

---

If

$$g[n] := x[\langle -n \rangle_N]$$

then

$$G[k] = X[\langle -k \rangle_N].$$

**Derivation:**

$$\begin{aligned} G[k] &= \sum_{n=0}^{N-1} x[\langle -n \rangle_N] W_N^{-nk} \\ &= \sum_{m=0}^{N-1} x[m] W_N^{-\langle -m \rangle_N k} \\ &= \sum_{m=0}^{N-1} x[m] W_N^{mk} \\ &= X[-k] \\ &= X[\langle -k \rangle_N] \end{aligned}$$

where we used the change of variables  $m = \langle -n \rangle_N$  (in which case  $n = \langle -m \rangle_N$  for  $0 \leq n \leq N - 1$ ).



## TIME-REVERSAL PROPERTY OF THE DFT

---

The following MATLAB code fragment illustrates the time-reversal property.

```
>> N = 5;
>> n = 0:N-1;
>> x = [3 6 2 4 7]';
>> X = fft(x);
>> G = X(mod(-n,N)+1);
>> g = ifft(G)
```

g =

```
3.0000 - 0.0000i
7.0000 + 0.0000i
4.0000 + 0.0000i
2.0000
6.0000 - 0.0000i
```

## COMPLEX CONJUGATION PROPERTY OF THE DFT

---

If

$$g[n] := x^*[n]$$

then

$$G[k] := X^*[\langle -k \rangle_N]$$

**Derivation:**

$$\begin{aligned} G[k] &= \sum_{n=0}^{N-1} x^*[n] W_N^{-nk} \\ &= \left( \sum_{n=0}^{N-1} x[n] W_N^{nk} \right)^* \\ &= X^*[-k] \\ &= X^*[\langle -k \rangle_N] \end{aligned}$$

## COMPLEX CONJUGATION PROPERTY OF THE DFT

---

The following MATLAB code fragment illustrates the complex conjugation property.

```
>> N = 5;
>> n = 0:N-1;
>> i = sqrt(-1);
>> x = [3+2*i 5-3*i 6+i 4-7*i 8+9*i].'
```

x =

```
3.0000 + 2.0000i
5.0000 - 3.0000i
6.0000 + 1.0000i
4.0000 - 7.0000i
8.0000 + 9.0000i
```

```
>> X = fft(x);
>> G = conj(X(mod(-n,N)+1));
>> g = ifft(G)
```

g =

```
3.0000 - 2.0000i
5.0000 + 3.0000i
6.0000 - 1.0000i
4.0000 + 7.0000i
8.0000 - 9.0000i
```

## REAL $x[n]$ PROPERTY OF THE DFT

---

$$x[n] \text{ real} \implies X[k] = X^*[\langle -k \rangle_N].$$

Equivalently,

$$x[n] \text{ real} \implies X_r[k] = X_r[\langle -k \rangle_N] \quad \text{and} \quad X_i[k] = -X_i[\langle -k \rangle_N].$$

where  $X_r[k]$  and  $X_i[k]$  are the real and imaginary parts of  $X[k]$ .

**Derivation:**

$$\begin{aligned} x[n] \text{ real} &\implies x[n] = x^*[n] \\ &\implies \text{DFT}\{x[n]\} = \text{DFT}\{x^*[n]\} \\ &\implies X[k] = X^*[\langle -k \rangle_N]. \end{aligned}$$

The following MATLAB code fragment illustrates the circular symmetries present in the DFT of a real signal. In this case, the signal is of odd length. Notice that  $X_i(0)$  must be 0 because when  $k$  is 0, the relation  $X[k] = X^*[\langle -k \rangle_N]$  gives  $X_i[0] = -X_i[0]$ .

```
>> x = [3 7 5 2 4 1 5]'
```

```
x =
```

```
3
7
5
2
4
1
5
```

```
>> X = fft(x)
```

```
X =
```

```
27.0000  
 3.7409 - 4.5956i  
-1.3351 - 1.7780i  
-5.4058 + 4.2094i  
-5.4058 - 4.2094i  
-1.3351 + 1.7780i  
 3.7409 + 4.5956i
```

```
>> [real(X) imag(X)]
```

```
ans =
```

```
27.0000      0  
 3.7409  -4.5956  
-1.3351  -1.7780  
-5.4058   4.2094  
-5.4058  -4.2094  
-1.3351   1.7780  
 3.7409   4.5956
```

In the following example, the signal is of even length. Notice that  $X_i(N/2) = 0$  in this case.

```
>> x = [3 7 5 2 4 1 3 2]'
```

```
x =
```

```
3  
7  
5
```

```
2
4
1
3
2
```

```
>> X = fft(x)
```

```
X =
```

```
27.0000
 3.2426 - 6.2426i
-1.0000 - 4.0000i
-5.2426 - 2.2426i
 3.0000
-5.2426 + 2.2426i
-1.0000 + 4.0000i
 3.2426 + 6.2426i
```

```
>> [real(X) imag(X)]
```

```
ans =
```

```
27.0000      0
 3.2426    -6.2426
-1.0000    -4.0000
-5.2426    -2.2426
 3.0000      0
-5.2426     2.2426
-1.0000     4.0000
 3.2426     6.2426
```

## Real and Circularly Symmetric $x[n]$

---

When  $x[n]$  is real and circularly symmetric, then so is the DFT.

$$\{x[n] \text{ real and } x[n] = x[\langle -n \rangle_N]\} \implies \{X[k] \text{ real and } X[k] = X[\langle -k \rangle_N]\}$$

**Derivation:**

$$x[n] \text{ real} \implies X[k] = X^*[\langle -k \rangle_N]$$

and

$$\begin{aligned} x[n] = x[\langle -n \rangle_N] &\implies \text{DFT}\{x[n]\} = \text{DFT}\{x[\langle -n \rangle_N]\} \\ &\implies X[k] = X[\langle -k \rangle_N]. \end{aligned}$$

Therefore

$$\begin{aligned} \{x[n] \text{ real and } x[n] = x[\langle -n \rangle_N]\} &\implies X[k] = X^*[\langle -k \rangle_N] = X[\langle -k \rangle_N] \\ &\implies X[k] \text{ real and } X[k] = X[\langle -k \rangle_N]. \end{aligned}$$

## Real and Circularly Symmetric $x[n]$

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The following example illustrates the DFT of a real circularly symmetric signal.

```
>> x = [3 1 5 4 6 4 5 1]'
```

```
x =
```

```
3  
1  
5  
4  
6  
4  
5  
1
```

```
>> X = fft(x)
```

```
X =
```

```
29.0000  
-7.2426 - 0.0000i  
-1.0000  
1.2426 - 0.0000i  
9.0000  
1.2426 + 0.0000i  
-1.0000  
-7.2426 + 0.0000i
```

Note that both  $x[n]$  and  $X[k]$  are real and circularly symmetric.



## SUMMARY OF DFT PROPERTIES

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Periodicity	$X[k] = X[\langle k \rangle_N]$	$x[n] = x[\langle n \rangle_N]$
Circular Shift	$x[\langle n - m \rangle_N]$	$W_N^{-mk} \cdot X[k]$
Modulation	$W_N^{mn} \cdot x[n]$	$X[\langle k - m \rangle_N]$
Circular Convolution	$x[n] \otimes g[n]$	$X[k] \cdot G[k]$
Time-Reversal	$x[\langle -n \rangle_N]$	$X[\langle -k \rangle_N]$
Complex Conjugation	$x^*[n]$	$X^*[\langle -k \rangle_N]$
Parseval's Theorem	$\sum_{n=0}^{N-1} x[n] \cdot g^*[n] = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X[k] \cdot G^*[k]$	