Pole-zero locations and frequency responses

September 2013 Ivan Selesnick

This note describes via animation the relationship between the pole-zero diagram and the magnitude response, for discrete-time LTI systems. The animations on the following pages require Adobe Reader. Other pdf viewers do not show the animations.

Z-Transform of $h(n)$

$$H(z) = \sum_n h(n) z^{-n} \tag{1}$$

DTFT of $h(n)$

$$H^f(\omega) = \sum_n h(n)\, \mathrm{e}^{-\mathrm{j}n\omega} \tag{2}$$

The unit circle is defined as:

$$\text{unit circle} = \{z \in \mathbb{C} \;:\; |z| = 1\} = \{\mathrm{e}^{\mathrm{j}\omega} \;:\; \omega \in [0,\, 2\pi]\} \tag{3}$$

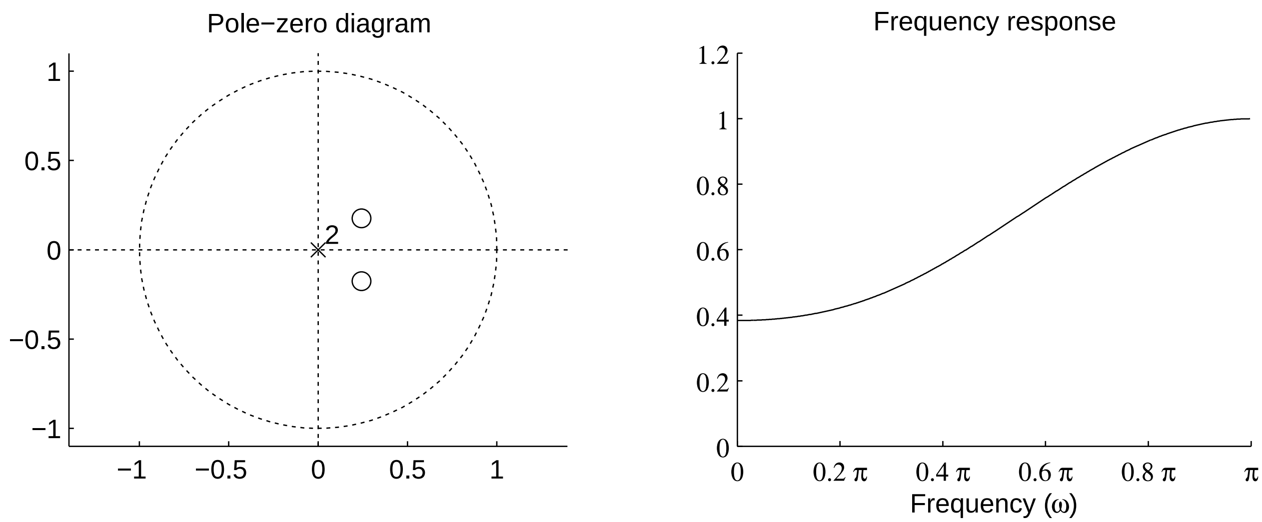
DTFT as evaluation of Z-transform on the unit circle

$$\boxed{H^f(\omega) = H(\mathrm{e}^{\mathrm{j}\omega})} \tag{4}$$

$H^f(\omega_1) = 0 \implies H(\mathrm{e}^{\mathrm{j}\omega_1}) = 0 \implies z_1 = \mathrm{e}^{\mathrm{j}\omega_1}$ is a zero of $H(z)$.

So, if the frequency response has a null at the frequency ω_1, then the transfer function has a zero on the unit circle at angle ω_1.

If the transfer function $H(z)$ has a zero *near* (but not on) the unit circle at angle ω_1, then $H^f(\omega_1) \approx 0$.



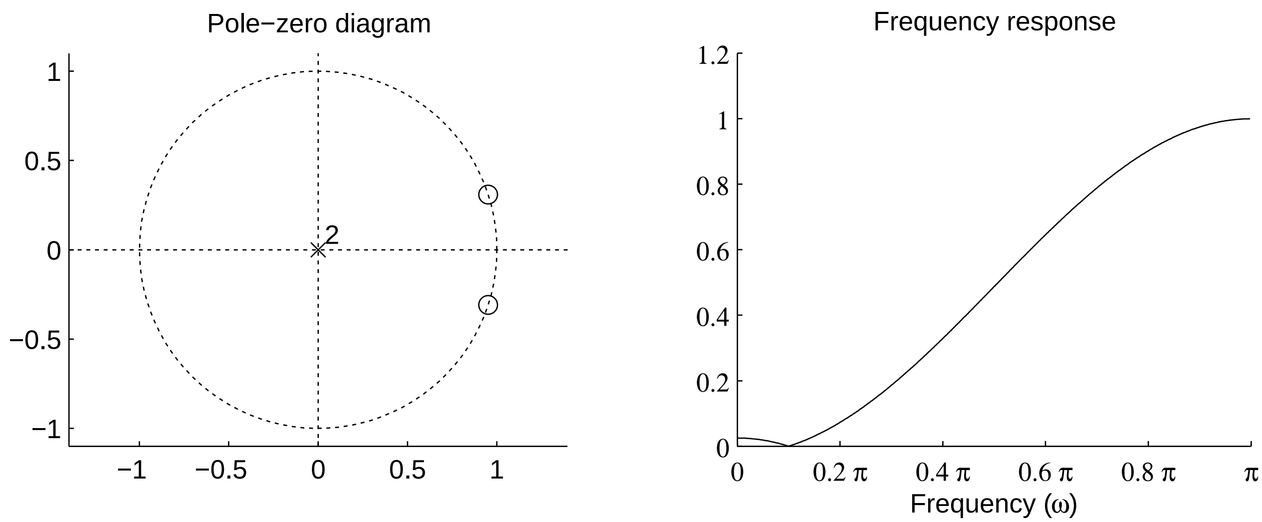
In this animation, $H(z)$ has a complex conjugate pair of zeros at $r\mathrm{e}^{\pm \mathrm{j}0.2\pi}$.

In the animation, the modulus r varies.

Observations:

1. When $r = 1$, the zeros are on the unit circle and the frequency response has nulls at $\omega = \pm 0.2\pi$.
2. When the zeros are close to the unit circle, the frequency response has dips at $\pm 0.2\pi$.
3. When the zeros are far from the unit circle, the frequency response is quite flat.

Zeros at the origin ($z = 0$) have no effect on $|H^f(\omega)|$.

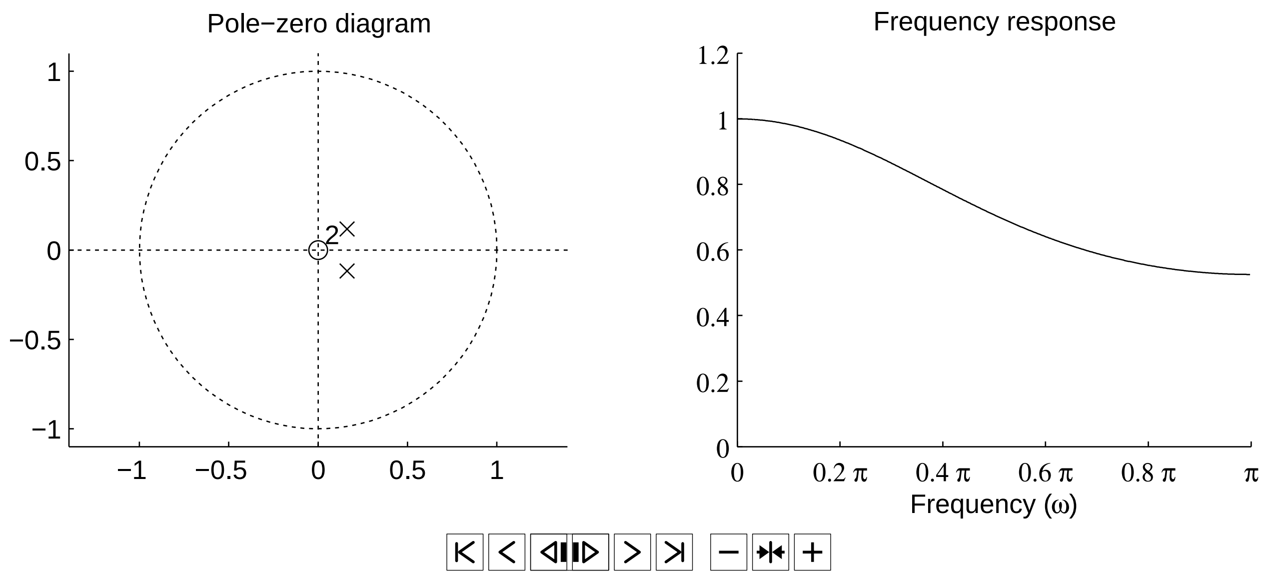


In this animation, $H(z)$ has a complex conjugate pair of zeros on the unit circle at $\mathrm{e}^{\pm \mathrm{j}\omega_1}$.

In the animation, the angle ω_1 varies.

Observations:

1. The zeros are on the unit circle, so the frequency response has nulls at frequencies $\pm\omega_1$.
2. As the zeros traverse the unit circle, the positions of the frequency response nulls vary.



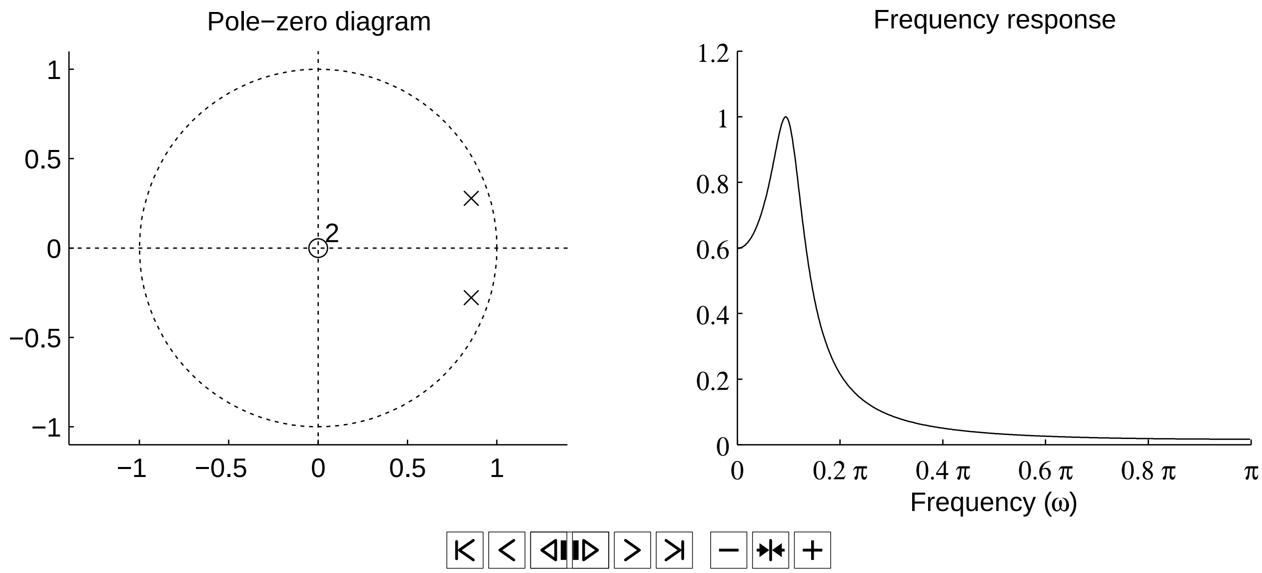
In this animation, $H(z)$ has a complex conjugate pair of poles at $r\mathrm{e}^{\pm \mathrm{j}0.2\pi}$.

In the animation, the modulus r varies.

Observations:

1. The poles must be strictly inside the unit circle for the system to be causal and stable.
2. When the poles are far from the unit circle, the frequency response is quite flat.
3. When the poles are close to the unit circle, the frequency response has peaks at $\pm 0.2\pi$.
4. The closer the poles are to the unit circle, the sharper the peak is.

Poles at the origin ($z = 0$) have no effect on $|H^f(\omega)|$.



In this animation, $H(z)$ has a complex conjugate pair of poles at $0.9\,\mathrm{e}^{\pm \mathrm{j}\omega_1}$.

In the animation, the angle ω_1 varies.

Observations:

1. The poles are close to the unit circle, so the frequency response has peaks at $\pm\omega_1$.
2. As the angle of the poles varies, the positions of the frequency response peaks vary.

What if an LTI system has poles and zeros away from the origin?

Then $H(z)$ can be written as $H(z) = H_1(z)\,H_2(z)$ where $H_1(z)$ and $H_2(z)$ have zeros and poles respectively away from the origin.

Further more, $H^f(\omega) = H_1^f(\omega)\,H_2^f(\omega)$, so $H^f(\omega)$ may have both peaks and dips (or nulls).