The Double-Density Dual-Tree Discrete Wavelet Transform

Ivan W. Selesnick

July 27, 2001

Electrical Engineering, Polytechnic University

6 Metrotech Center, Brooklyn, NY 11201

tel: 718 260-3416, fax: 718 260-3906

selesi@taco.poly.edu

EDICS: SP.2-MWAV

Multirate Processing and Wavelets

Abstract

This paper introduces the double-density dual-tree DWT, a discrete wavelet transform that combines the double-density DWT and the dual-tree DWT, each of which has its own characteristics and advantages. The transform corresponds to a new family of dyadic wavelet tight frames based on two scaling functions and four distinct wavelets, $\psi_{h,i}(t)$ and $\psi_{g,i}(t)$, $i = 1, 2$. The two wavelets $\psi_{h,1}(t)$ and $\psi_{h,2}(t)$ are off-set from one another by one half, $\psi_{h,1}(t) \approx \psi_{h,2}(t-0.5)$ and $\psi_{g,i}(t)$ likewise. Therefore, the integer translates of one wavelet pair fall midway between the integer translates of the other pair. Simultaneously, the two wavelets $\psi_{g,1}(t)$ and $\psi_{h,1}(t)$ form an approximate Hilbert transform pair, $\psi_{g,1}(t) \approx \mathcal{H}\{\psi_{h,1}(t)\}$, and likewise $\psi_{g,2}(t) \approx \mathcal{H}\{\psi_{h,2}(t)\}$. Therefore they can be used to implement complex and directional wavelet transforms. The paper develops a design procedure to obtain FIR filters which satisfy the numerous constraints imposed. This design procedure employs a fractional-delay all-pass filter, spectral factorization, and filter bank completion. The solutions have vanishing moments, compact support, a high degree of smoothness, and are nearly shift-invariant.

Permission to publish this abstract separately is granted.

Submitted to *IEEE Transactions on Signal Processing*.

1 Introduction

Recently, research on two distinct types of wavelet frames has emerged, each with its own characteristics and advantages.

The first type of wavelet frame consists simply of the concatenation of two wavelet bases, where the wavelets are designed to work together to achieve specific benefits. In particular, Kingsbury has demonstrated that dramatic improvements can be obtained in wavelet-based signal processing by utilizing a pair of wavelet transforms where the wavelets form a Hilbert transform pair, $\psi_g(t) = \mathcal{H}\{\psi_h(t)\}$. Kingsbury calls a wavelet transform of this type a *dual-tree* discrete wavelet transform (DWT) [8, 9, 10]. The dual-tree DWT has several advantages, including (i) near shift-invariance; (ii) significantly improved denoising capability; and (iii) the implementation of directional 2D DWTs using separable filter banks. The advantages of a Hilbert pair of wavelet bases have also been recognized by other authors, for example, Abry and Flandrin [1, 2]; Freeman, Adelson, and Simoncelli [7, 23, 24]. The question of how to design the two lowpass scaling filters so that the two wavelets form an approximate Hilbert transform pair was recently addressed by the author in [20] where a simple characterization is given. A design procedure based on that result is described in [18, 19]. (The design procedure employs spectral factorization and the flat-delay filter.)

A second type of wavelet frame is based on over-sampled filter banks, the advantages of which are clearly demonstrated in [24] which develops *shiftable multiscale transforms*. In [17, 21, 22] we considered the design of wavelet tight frames, based on a single scaling function and two distinct wavelets, that are analogous to Daubechies' orthonormal wavelet bases — that is, the design of wavelet filters of minimal length satisfying certain polynomial properties, but now in the oversampled case. Wavelet frames of this general type (also known as *affine* frames) have also been discussed by Ron and Shen [15, 16], Chui and He [3], and Petukhov [12, 13]. (More recent developments are reported in [4, 5].) The advantages of this type of wavelet frame include (i) very smooth wavelets with short support (because there are more degrees of freedom in the design problem); (ii) near shift invariance; (iii) improved time-frequency bandwidth product; and (iv) approximation to the continuous wavelet transform (having more wavelets than necessary gives a closer spacing between adjacent wavelets within the same scale). The wavelets can be constructed using maximally flat FIR filters in conjunction with spectral factorization and extension methods for paraunitary matrices. In particular, the wavelets can be designed so that they are off-set from one another by one half — the integer translates of one wavelet fall midway between the integer translates of the other wavelet. Hence the term *double-density* DWT. Originally Gröbner bases were used to obtain these wavelets in [21, 22], but the paper by Chui and He [3] on the design of wavelet frames of similar character since made clear how to apply the paraunitary method which greatly simplifies the design procedure. An alternative simplified procedure is also described by Petukhov [12].

While these two types of wavelet frames are similar in some respects (they are both nearly shift-invariant and they are both redundant by a factor of 2 in 1D), they also have basic differences. (*i*) In the dual-tree DWT the two wavelets form an approximate Hilbert transform pair, while in the double-density DWT the two wavelets are off-set by one half. (*ii*) For the dual-tree DWT there are fewer degrees of freedom for design (achieving the Hilbert pair property adds constraints), while for the double-density DWT there are more degrees of freedom for design. (*iii*) The dual-tree and double-density DWTs are implemented with totally different filter bank structures. (*iv*) The dual-tree DWT can be interpreted as a complex-valued wavelet transform which is useful for signal modeling and denoising (the double-density DWT can not be interpreted as such). (*iv*) The dual-tree DWT can be used to implement two-dimensional transforms with directional wavelets, which is highly desirable for image processing (the double-density DWT can not be, but it can be used in conjunction with specialized post-filters to implement a complex wavelet transform with low-redundancy as developed in [6]).

In this paper we present (approximate) Hilbert transform pairs of wavelet frames that have the advantages of both types of wavelet frames described above. The DWT based on these wavelet frames can then be called the *double-density dual-tree DWT*. The design procedure is based on the flat-delay filter, spectral factorization, and paraunitary extension. There are four wavelets,

$$\psi_{h,i}(t), \quad \psi_{g,i}(t), \quad i = 1, 2,$$

where

$$\psi_{g,1}(t) \approx \mathcal{H}\{\psi_{h,1}(t)\}, \qquad \psi_{g,2}(t) \approx \mathcal{H}\{\psi_{h,2}(t)\} \tag{1}$$

and

$$\psi_{h,1}(t) \approx \psi_{h,2}(t - 0.5), \qquad \psi_{g,1}(t) \approx \psi_{g,2}(t - 0.5) \tag{2}$$

The two wavelets $\psi_{h,1}(t)$ and $\psi_{h,2}(t)$ are off-set from one another by one half, and $\psi_{g,i}(t)$ likewise. The solutions have vanishing moments and compact support. They are much smoother than the dual-tree wavelets and unlike the double-density wavelets, they form approximate Hilbert transform pairs. As they combine the advantages of the double-density DWT and the dual-tree DWT, it is expected that they will be useful in several signal processing applications.

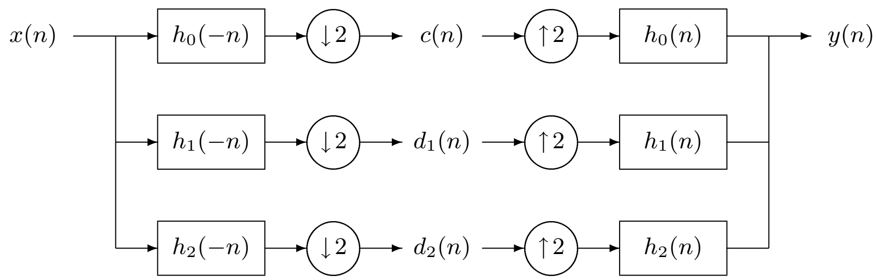

Figure 1: An oversampled analysis and synthesis filter bank.

2 Preliminaries

The dual-tree DWT is based on concatenating two critically sampled DWTs. The filter bank structure corresponding to the dual-tree DWT simply consists of two critically sampled iterated filter banks operating in parallel. The performance gains provided by the dual-tree DWT come from designing the filters in the two filter banks appropriately.

The double-density dual-tree DWT proposed in this paper is based on concatenating two oversampled DWTs. The filter bank structure corresponding to the double-density dual-tree DWT consists of two oversampled iterated filter banks operating in parallel, similar to the dual-tree DWT. The oversampled filter bank is illustrated in Figure 1. We will denote the filters in the first filter bank by $h_i(n)$ and the filters in the second filter bank by $g_i(n)$, for $i = 0, 1, 2$. Note that in each of the filter banks to be considered in this paper, the synthesis filters are the time-reversed versions of the analysis filters. The goal will be to design the 6 FIR filters so that (*i*) they satisfy the perfect reconstruction property, (*ii*) the wavelets form two (approximate) Hilbert transform pairs, (*iii*) the wavelets have specified vanishing moments, (*iv*) the filters are of short support.

The Z-transform of $h_i(n)$ is denoted by $H_i(z)$,

$$H_i(z) = \mathrm{ZT}\{h_i(n)\} := \sum_n h_i(n)\, z^{-n},$$

$$G_i(z) = \mathrm{ZT}\{g_i(n)\} := \sum_n g_i(n)\, z^{-n}.$$

Through the paper, it is assumed that all filter coefficients $h_i(n)$, $g_i(n)$ are real valued. The frequency response of the filters are given by $H_i(e^{j\omega})$ and $G_i(e^{j\omega})$,

$$H_i(e^{j\omega}) = \mathrm{DTFT}\{h_i(n)\} := \sum_n h_i(n)\, e^{-j\,n\,\omega},$$
$$G_i(e^{j\omega}) = \mathrm{DTFT}\{g_i(n)\} := \sum_n g_i(n)\, e^{-j\,n\,\omega}.$$

The filters $h_i(n)$ and $g_i(n)$ should satisfy the perfect reconstruction (PR) conditions. From basic multirate identities, the PR conditions are the following.

$$H_0(z)\, H_0(1/z) + H_1(z)\, H_1(1/z) + H_2(z)\, H_2(1/z) = 2, \tag{3}$$
$$H_0(z)\, H_0(-1/z) + H_1(z)\, H_1(-1/z) + H_2(z)\, H_2(-1/z) = 0, \tag{4}$$

and

$$G_0(z)\, G_0(1/z) + G_1(z)\, G_1(1/z) + G_2(z)\, G_2(1/z) = 2, \tag{5}$$
$$G_0(z)\, G_0(-1/z) + G_1(z)\, G_1(-1/z) + G_2(z)\, G_2(-1/z) = 0. \tag{6}$$

The scaling and wavelet functions are defined implicitly through the dilation and wavelet equations:

$$\phi_h(t) = \sqrt{2} \sum_n h_0(n)\, \phi_h(2t-n)$$
$$\psi_{h,1}(t) = \sqrt{2} \sum_n h_1(n)\, \phi_h(2t-n)$$
$$\psi_{h,2}(t) = \sqrt{2} \sum_n h_2(n)\, \phi_h(2t-n)$$

and

$$\phi_g(t) = \sqrt{2} \sum_n g_0(n)\, \phi_g(2t-n)$$
$$\psi_{g,1}(t) = \sqrt{2} \sum_n g_1(n)\, \phi_g(2t-n)$$
$$\psi_{g,2}(t) = \sqrt{2} \sum_n g_2(n)\, \phi_g(2t-n)$$

The Fourier transforms of the scaling functions and wavelets will be denoted as

$$\Phi_h(\omega) = \mathcal{F}\{\phi_h(t)\},$$
$$\Phi_g(\omega) = \mathcal{F}\{\phi_g(t)\},$$
$$\Psi_{h,i}(\omega) = \mathcal{F}\{\psi_{h,i}(t)\},$$
$$\Psi_{g,i}(\omega) = \mathcal{F}\{\psi_{g,i}(t)\}.$$

The Hilbert transform of a function $f(t)$ will be denoted as $\mathcal{H}\{f(t)\}$. Following the work by Kingsbury, we want the wavelets to form Hilbert transform pairs,

$$\psi_{g,1}(t) = \mathcal{H}\{\psi_{h,1}(t)\}, \tag{7}$$

$$\psi_{g,2}(t) = \mathcal{H}\{\psi_{h,2}(t)\}. \tag{8}$$

Recalling the definition of the Hilbert transform, this means that

$$\Psi_{g,i}(\omega) = \begin{cases} -j\ \Psi_{h,i}(\omega), & \omega > 0 \\ \ \ j\ \Psi_{h,i}(\omega), & \omega < 0. \end{cases} \tag{9}$$

3 Filter Constraints for Hilbert Pair Property

Because the DWT is implemented using discrete-time filter banks and because the discrete-time filters $h_i(n)$ and $g_i(n)$ define the wavelets, it is necessary to translate the Hilbert transform relations (7,8) into constraints to be imposed on the filter bank. That is, what constraints should the filters $h_i(n)$ and $g_i(n)$ satisfy so that $\psi_{h,i}(t)$ and $\psi_{g,i}(t)$ form a Hilbert pair? That is the question addressed in this section.

If $\psi_{h,i}(t)$ and $\psi_{g,i}(t)$ form a Hilbert transform pair, then

$$|\Psi_{h,i}(\omega)| = |\Psi_{g,i}(\omega)|.$$

This suggests, from the infinite product formula, that we should have

$$|G_i(e^{j\omega})| = |H_i(e^{j\omega})|, \quad i = 0, 1, 2.$$

That is, the filters should be related as

$$G_i(e^{j\omega}) = H_i(e^{j\omega})\ e^{-j\theta_i(\omega)} \tag{10}$$

where $\theta_i(\omega)$ are 2π-periodic. But how should the three phase functions $\theta_i(\omega)$ be selected so as to have $\psi_{g,i}(t) = \mathcal{H}\{\psi_{h,i}(t)\}$?

The characterization of Hilbert pairs of wavelet bases in [20] involved a similar problem, however, in that case the highpass filter $H_1(z)$ is fully determined by the lowpass filter $H_0(z)$ and there was no filter $H_2(z)$. In the oversampled filter bank considered here, the highpass filters $H_1(z)$ and $H_2(z)$ are not fully determined by the lowpass filter $H_0(z)$, however, the approach introduced in [20] carries over as follows.

From the infinite product formula we can derive as in [20] the following expression relating $\Phi_g(\omega)$ and $\Phi_h(\omega)$,

$$\Phi_g(\omega) = \Phi_h(\omega)\, e^{-j \sum_{k=1}^{\infty} \theta_0(\omega/2^k)}.$$

Similarly, we can derive the following expression relating $\Psi_{g,i}(\omega)$ and $\Psi_{h,i}(\omega)$,

$$\Psi_{g,i}(\omega) = \Psi_{h,i}(\omega)\, e^{-j\left[\theta_i(\omega/2) + \sum_{k=2}^{\infty} \theta_0(\omega/2^k)\right]}, \quad i = 1, 2. \tag{11}$$

From (11) and (9) we see that $\theta_i(\omega)$ must satisfy the following condition.

$$\theta_i(\omega/2) + \sum_{k=2}^{\infty} \theta_0(\omega/2^k) = \begin{cases} 0.5\,\pi, & \omega > 0 \\ -0.5\,\pi, & \omega < 0 \end{cases}, \quad i = 1, 2 \tag{12}$$

Following the derivation in [20] for orthogonal wavelet bases, we find that for the wavelet frames considered here, if the 2π-periodic functions $\theta_i(\omega)$ are defined as

$$\theta_0(\omega) = \frac{\omega}{2}, \qquad |\omega| < \pi \tag{13}$$

and

$$\theta_i(\omega) = -\theta_0(\omega - \pi), \quad i = 1, 2 \tag{14}$$

then condition (12) holds and therefore $\{\psi_{h,i}(t),\ \psi_{g,i}(t)\}$ form a Hilbert transform pair for $i = 1, 2$.

4 Filter Design for the Double-density Dual-tree DWT

The design problem can be stated as follows. Given K_i, $i = 0, 1, 2$, find 6 FIR filters $h_i(n)$, $g_i(n)$ of short support satisfying:

1. The perfect reconstruction conditions (3-6).

2. The Hilbert pair relations (1) or equivalently, the conditions (13,14) should be nearly satisfied.

3. Zero moment conditions,

$$\int t^k\, \psi_{h,i}(t)\, dt = \int t^k\, \psi_{g,i}(t)\, dt = 0, \quad i = 1, 2$$

   for $0 \le k \le K_i - 1$, where K_1, K_2 are specified.

4. Lowpass zeros at $\omega = \pi$.

$$(1 + z^{-1})^{K_0} \mid H_0(z) \quad \text{and} \quad (1 + z^{-1})^{K_0} \mid G_0(z).$$

5. The shift property (2).

As discussed in [17], by making $K_0 > K_1$, $K_0 > K_2$ we can obtain wavelets with a high degree of smoothness which also have the shift property (2). As in [17], it is unnecessary to to specify any explicit constraint to ensure that $\psi_{h,1}(t) \approx \psi_{h,2}(t-0.5)$; it is sufficient that K_0 exceed K_1 and K_2. Then, by completing the filter bank as described below and in [17], it is possible to obtain two wavelets where one wavelet is similar to the other translated by one half.

The design procedure for the double-density dual-tree DWT introduced here draws on the design procedures for the double-density DWT and the dual-tree DWT described in [3, 17] and [18, 19]. We propose for the double-density dual-tree DWT that the set of filters take the following form,

$$H_0(z) = D(z)\,(1+z^{-1})^{K_0}\,Q_0(z) \tag{15}$$
$$H_1(z) = (-z)^{-L}\,D(-1/z)\,(1-z^{-1})^{K_1}\,Q_1(z) \tag{16}$$
$$H_2(z) = (-z)^{-L}\,D(-1/z)\,(1-z^{-1})^{K_2}\,Q_2(z) \tag{17}$$
$$G_0(z) = z^{-L}\,D(1/z)\,(1+z^{-1})^{K_0}\,Q_0(z) \tag{18}$$
$$G_1(z) = D(-z)\,(1-z^{-1})^{K_1}\,Q_1(z) \tag{19}$$
$$G_2(z) = D(-z)\,(1-z^{-1})^{K_2}\,Q_2(z). \tag{20}$$

Equivalently,

$$h_0(n) = d(n) * s_0(n) * q_0(n) \tag{21}$$
$$h_1(n) = (-1)^n\,d(L-n) * s_1(n) * q_1(n) \tag{22}$$
$$h_2(n) = (-1)^n\,d(L-n) * s_2(n) * q_2(n) \tag{23}$$
$$g_0(n) = d(L-n) * s_0(n) * q_0(n) \tag{24}$$
$$g_1(n) = (-1)^n\,d(n) * s_1(n) * q_1(n) \tag{25}$$
$$g_2(n) = (-1)^n\,d(n) * s_2(n) * q_2(n) \tag{26}$$

where

$$s_0(n) = \binom{K_0}{n} = \frac{K_0!}{(K_0-n)!\,n!}$$

$$s_1(n) = (-1)^n \binom{K_1}{n}$$

$$s_2(n) = (-1)^n \binom{K_2}{n}.$$

These expressions incorporate the moment properties. There are four transfer functions, $D(z)$ and $Q_i(z)$, $i = 0, 1, 2$, to be determined according to the remaining properties (PR conditions and approximate Hilbert pair conditions). It will be illustrated in the examples below, that a wavelet frame with the sought properties can be obtained with a set of filters taking the above form. $D(z)$ will be determined first. $D(z)$ will be determined so that the Hilbert pair relations are approximately satisfied. $Q_0(z)$ will be determined second; then the two lowpass filters $H_0(z)$ and $G_0(z)$ will be known. $Q_1(z)$ and $Q_2(z)$ will then be obtained by filter bank completion (paraunitary extension) as described in Section 5.

From (15) and (18) we can write

$$G_0(z) = H_0(z)\,\frac{z^{-L} D(1/z)}{D(z)} \tag{27}$$

where we can recognize that the transfer function

$$A(z) := \frac{z^{-L} D(1/z)}{D(z)} \tag{28}$$

is an all-pass system, $|A(e^{j\omega})| = 1$, or $A(e^{j\omega}) = e^{-j\theta_a(\omega)}$ for some 2π-periodic phase function $\theta_a(\omega)$. Then, from (27),

$$G_0(e^{j\omega}) = H_0(e^{j\omega})\; e^{-j\theta_a(\omega)},$$

so, referring to (10), we have $\theta_0(\omega) = \theta_a(\omega)$. The phase difference $\theta_0(\omega)$ is given by the phase function of the all-pass filter $A(z)$. Similarly, from (16,17) and (19,20) we can write

$$G_i(z) = A(-1/z)\, H_i(z), \quad i = 1, 2$$

where the transfer function $A(-1/z)$ is also all-pass. We have, for $i = 1, 2$,

$$\begin{aligned} G_i(e^{j\omega}) &= A(-e^{-j\omega})\; H_i(e^{j\omega}) \\ &= \overline{A(e^{j(\omega-\pi)})}\; H_i(e^{j\omega}) \\ &= e^{j\theta_a(\omega-\pi)}\; H_i(e^{j\omega}). \end{aligned}$$

That is, $\theta_i(\omega) = -\theta_a(\omega - \pi)$ for $i = 1, 2$.

Therefore, with filters having the forms proposed in (15-20), the phase difference $\theta_0(\omega)$ is identical to the phase function $\theta_a(\omega)$ of the all-pass system $A(z)$, while for $i = 1, 2$, the phase difference $\theta_i(\omega)$ is given by $-\theta_a(\omega - \pi)$. From equations (13) and (14), it follows directly that $D(z)$ should be chosen so that the phase function $\theta_a(\omega)$ of the all-pass system is $\omega/2$ for $|\omega| < \pi$. Because the phase function of $A(z)$ can not equal $\omega/2$ exactly when $D(z)$ is of finite degree, it is necessary to employ an approximation. An all-pass system whose phase function approximates $\tau\,\omega$ for some range of ω is simply a fractional-delay all-pass filter (see [11] for a review of such systems). This type of system delays its input signal by τ samples.

We will take $A(z)$ to be a maximally-flat fractional-delay all-pass filter. (Any fractional-delay all-pass filter could be used here.) The maximally-flat type, which is accurate with a degree (L) of tangency at $\omega = 0$, is given by the following formula, adapted from [25].

$$D(z) = 1 + \sum_{n=1}^{L} d(n)\, z^{-n}$$

with

$$d(n) = (-1)^n \binom{L}{n} \frac{(\tau - L)_n}{(\tau + 1)_n} \tag{29}$$

where $(x)_n$ represents the rising factorial,

$$(x)_n := \underbrace{(x)\,(x+1)\,\cdots\,(x+n-1)}_{n \text{ terms}}.$$

With this $D(z)$ we have the approximation

$$\begin{aligned} A(z) &\approx z^{-\tau} \quad \text{around } z = 1, \\ A(e^{j\omega}) &\approx e^{-j\tau\omega} \quad \text{around } \omega = 0, \end{aligned}$$

or equivalently,

$$\theta_a(\omega) \approx \tau\,\omega \quad \text{around } \omega = 0.$$

The coefficients $d(n)$ in (29) can be computed very efficiently with the following recursion,

$$\begin{aligned} d(0) &= 1 \\ d(n+1) &= d(n) \cdot \frac{(L-n)(L-n-\tau)}{(n+1)(n+1+\tau)}, \quad 0 \le n \le L-1. \end{aligned}$$

In our problem, we will use $d(n)$ in (29) with $\tau = 1/2$. For example, with $L = 1$, we have

$$d(n) = \{1,\ 1/3\}\,, \quad \text{for} \quad n = 0, 1.$$

With $L = 2$, we have

$$d(n) = \{1,\ 2,\ 1/5\}\,, \quad \text{for} \quad n = 0, 1, 2.$$

With $L = 3$, we have

$$d(n) = \{1,\ 5,\ 3,\ 1/7\}, \quad \text{for} \quad n = 0, 1, 2, 3.$$

The phase function $\theta_a(\omega)$ of the maximally-flat fractional-delay all-pass filter $A(z)$ in (28) with these $D(z)$ are shown in Figure 2. For larger values of L an improved approximation to $\omega/2$ is obtained. The line $\omega/2$ is indicated in the figure by the dashed line. Note that the behavior of $\theta_a(\omega)$ is not important in the stop-band of the lowpass filter $H_0(z)$, so the deviation of $\theta_a(\omega)$ from $\omega/2$ near $\omega = \pi$ is not relevant.

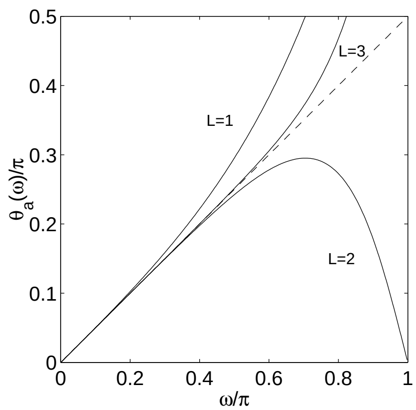

Figure 2: The phase function $\theta_a(\omega)$ of the maximally-flat fractional-delay all-pass filter with $\tau = 0.5$ and $L = 1, 2, 3$.

With $D(z)$ chosen as described here, the filters (15-20) automatically satisfy (approximately) the Hilbert pair property and have the specified number of vanishing moments. The remaining design problem is to find $Q_i(z)$ (of minimal degree) so that the perfect reconstructions conditions are satisfied. Note that $h_i(n)$ and $g_i(n)$ have the same autocorrelation function,

$$H_i(z)\, H_i(1/z) = G_i(z)\, G_i(1/z), \quad i = 0, 1, 2.$$

Therefore, if the PR conditions (3) are satisfied then the PR conditions (5) are also satisfied. Also note that

$$G_i(z)\, G_i(-1/z) = A(z)\, A(-1/z)\, H_i(z)\, H_i(1/z)$$

for $i = 0, 1, 2$ and in turn

$$\begin{aligned} &G_0(z)\, G_0(-1/z) + G_1(z)\, G_1(-1/z) + G_2(z)\, G_2(-1/z) \\ &= A(z)\, A(-1/z)\, \left[H_0(z)\, H_0(-1/z) + H_1(z)\, H_1(-1/z) + H_2(z)\, H_2(-1/z)\right]. \end{aligned}$$

Therefore, if the PR conditions (4) are satisfied, then the PR conditions (6) are also satisfied. So we only need to concentrate on the conditions (3) and (4).

Let's define the product filter $P_0(z)$ as

$$P_0(z) := H_0(z)\, H_0(1/z) = G_0(z)\, G_0(1/z) \tag{30}$$

$$= D(z)\, D(1/z)\, (z + 2 + z^{-1})^{K_0}\, Q_0(z)\, Q_0(1/z) \tag{31}$$

$$= D(z)\, D(1/z)\, (z + 2 + z^{-1})^{K_0}\, R_0(z), \tag{32}$$

where

$$R_0(z) := Q_0(z)\, Q_0(1/z), \tag{33}$$

and for $i = 1, 2$ lets define the product filter $P_i(z)$ as

$$P_i(z) := H_i(z)\, H_i(1/z) = G_i(z)\, G_i(1/z) \tag{34}$$

$$= D(-z)\, D(-1/z)\, (-z + 2 - z^{-1})^{K_i}\, Q_i(z)\, Q_i(1/z). \tag{35}$$

Then the perfect reconstruction conditions (3,5) can be written as

$$P_0(z) + P_1(z) + P_2(z) = 2. \tag{36}$$

In our examples we will ask that the four wavelets have the same number of vanishing moments (then neither wavelet does more 'work' than the other). That is, we will use $K_1 = K_2$ in the examples. Because this also simplifies the notation in the following derivation, in the remainder of the paper we assume that $K_1 = K_2$. We also assume that $K_0 > K_1$. If we define

$$R_{12}(z) := Q_1(z)\, Q_1(1/z) + Q_2(z)\, Q_2(1/z),$$

then we can write

$$P_1(z) + P_2(z) = D(-z)\, D(-1/z)\, (-z + 2 - z^{-1})^{K_1}\, R_{12}(z). \tag{37}$$

Combining (32) and (37), the PR conditions (3, 5) are written as

$$2 = D(z)\, D(1/z)\, (z + 2 + z^{-1})^{K_0}\, R_0(z) + D(-z)\, D(-1/z)\, (-z + 2 - z^{-1})^{K_1}\, R_{12}(z)$$

where $r_0(n) = r_0(-n)$ and $r_{12}(n) = r_{12}(-n)$. Then, given K_0, K_1 and $D(z)$, we can find $R_0(z)$ and $R_{12}(z)$ straight forwardly by solving a linear system of equations or by using the extended Euclidean algorithm for GCD computations. From the Euclidean algorithm, we know in advance that $r_0(n)$ will be of length $2L+2K_1-1$ and that $r_{12}(n)$ will be of length $2L+2K_0-1$. The sequence $r_0(n)$ is supported on $1-L-K_1 \leq n \leq L+K_1-1$ and $r_{12}(n)$ is supported on $1-L-K_0 \leq n \leq L+K_0-1$. Once $R_0(z)$ is found, then $Q_0(z)$ can be obtained through spectral factorization from (33), which in turn gives $H_0(z)$ and $G_0(z)$. The length of $q_0(n)$ will be $L+K_1$. Note that $Q_0(z)$ is not unique, so neither is $H_0(z)$ nor $G_0(z)$. In the examples below, we take $Q_0(z)$ to be minimum-phase (all zeros of $Q_0(z)$ lie inside $|z|=1$). The length of $h_0(n)$ and $g_0(n)$ will be K_0+K_1+2L.

For example, when we have $K_0=4$, $K_1=K_2=2$, $L=2$, then the (minimum-phase) sequence $q_0(z)$ we obtain by this procedure is tabulated in Table 1. The lowpass filters $h_0(n)$ and $g_0(n)$, both of length 10, are illustrated in Figure 3, along with the frequency response and zero diagram of each. Note that $|H_0(e^{j\omega})| = |G_0(e^{j\omega})|$. The two zeros of $H_0(z)$ on the negative real axis are from $D(z)$, while the two zeros of $G_0(z)$ on the negative real axis are from and $D(1/z)$. The remaining zeros inside the unit circle are from $Q_0(z)$. Note that even though $Q_0(z)$ is minimum-phase, $H_0(z)$ and $G_0(z)$ are not, due to the factors $D(z)$ and $D(1/z)$ in $H_0(z)$ and $G_0(z)$ respectively.

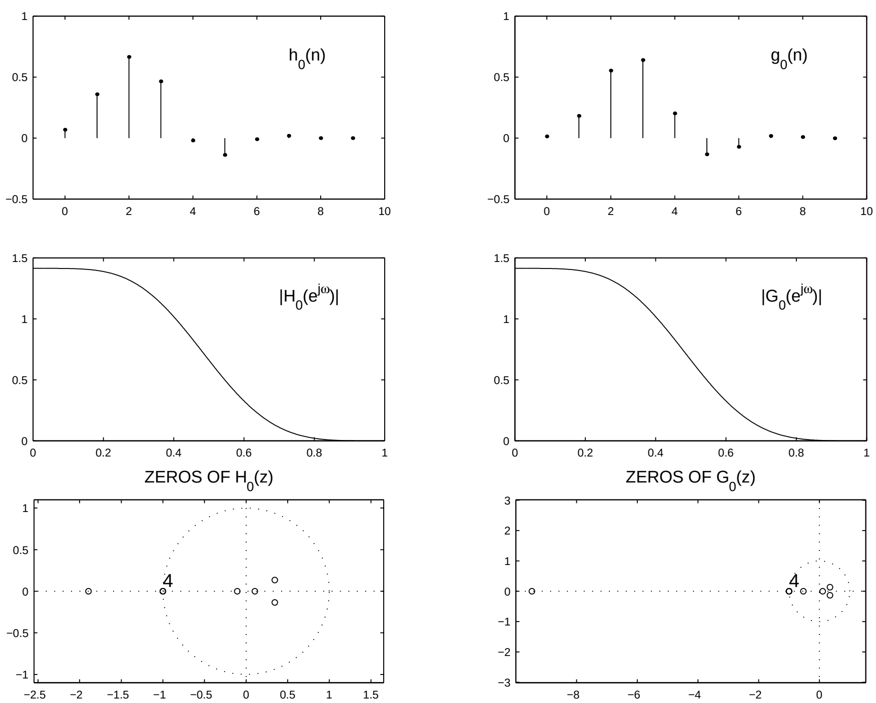

Figure 3: The lowpass filters $h_0(n)$ and $g_0(n)$ obtained with $K_0=4$, $K_1=K_2=2$, $L=2$. The filter coefficients are tabulated in Table 2.

n	$q_0(n)$
0	0.06911582051268
1	-0.05503365268588
2	0.01454236721253
3	-0.00100317639923

Table 1: The coefficients of (minimum-phase) $q_0(n)$ when $K_0 = 4, K_1 = K_2 = 2, L = 2$.

5 Constructing the Wavelet Filters

Once the low-pass filters $h_0(n)$ and $g_0(n)$ are obtained, the four (non-unique) wavelet filters $h_1(n)$, $h_2(n)$, $g_1(n)$, and $g_2(n)$ can be obtained using a polyphase formulation. We will concentrate on obtaining the filters $h_i(n)$, for then the filters $g_i(n)$ follow from (19,20). Define the polyphase components $H_{i0}(z)$ and $H_{i1}(z)$ through

$$H_i(z) = H_{i0}(z^2) + z^{-1} H_{i1}(z^2), \qquad i = 0, 1, 2 \tag{38}$$

and define the polyphase matrix $H(z)$ as

$$H(z) = \begin{bmatrix} H_{00}(z) & H_{10}(z) & H_{20}(z) \\ H_{01}(z) & H_{11}(z) & H_{21}(z) \end{bmatrix}.$$

Then the three-channel filter bank of Figure 1 can be redrawn as the filter bank of Figure 4. Similarly, the perfect reconstruction condition can be written as

$$H(z)\, H^t(1/z) = I_2. \tag{39}$$

The matrix $H(z)$ is said to be a 2×3 *lossless* system [26]. Once we find four components H_{10}, H_{11}, H_{20} and H_{21} so that $H(z)$ satisfies (39) we can then form $h_1(n)$ and $h_2(n)$.

One way to obtain a 2×3 lossless system is to first determine a 3×3 lossless system and to then delete the last row. Define $\hat{H}(z)$ to be the matrix

$$\hat{H}(z) = \begin{bmatrix} H_{00}(z) & H_{10}(z) & H_{20}(z) \\ H_{01}(z) & H_{11}(z) & H_{21}(z) \\ H_{02}(z) & H_{12}(z) & H_{22}(z) \end{bmatrix}$$

where only $H_{00}(z)$ and $H_{01}(z)$ are so far determined. We will design the square lossless system $\hat{H}(z)$, or *paraunitary matrix*, to satisfy

$$\hat{H}^t(1/z)\, \hat{H}(z) = \hat{H}(z)\, \hat{H}^t(1/z) = I_3. \tag{40}$$

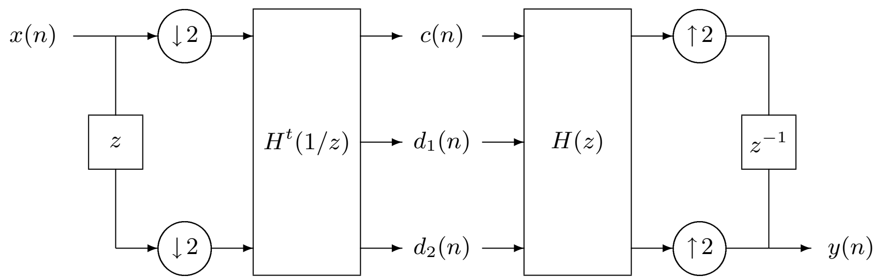

Figure 4: The oversampled filter bank of Figure 1 in polyphase form.

Then

$$H_{00}(z)\, H_{00}(1/z) + H_{01}(z)\, H_{01}(1/z) + H_{02}(z)\, H_{02}(1/z) = 1,$$

or

$$H_{02}(z)\, H_{02}(1/z) = 1 - H_{00}(z)\, H_{00}(1/z) - H_{01}(z)\, H_{01}(1/z). \tag{41}$$

Therefore $H_{02}(z)$ can be obtained by spectral factorization,

$$|H_{02}(e^{j\omega})|^2 = 1 - |H_{00}(e^{j\omega})|^2 - |H_{01}(e^{j\omega})|^2. \tag{42}$$

Note that $H_{02}(z)$ is not uniquely defined.

It turns out that the choice of $H_{02}(z)$ must be done here with care — not just any spectral factor of the right hand side of (42) will lead in what follows to filters $H_1(z)$, $H_2(z)$ having the form set out in (16) and (17). It is shown in the appendix that the transfer function $H_{02}(z)$ in (42) must satisfy the following conditions.

$$D(z)\, D(-z) \mid H_{02}(z^2)\, H_{02}(1/z^2) \tag{43}$$

$$D(z)\, D(-z) \mid H_{02}(z^2)\, H_{12}(1/z^2) \tag{44}$$

$$D(z)\, D(-z) \mid H_{02}(z^2)\, H_{22}(1/z^2) \tag{45}$$

This suggests that $H_{02}(z)$ should satisfy

$$D(z)\, D(-z) \mid H_{02}(z^2)$$

which was verified numerically in examples to be the appropriate constraint on the spectral factorization of (42). Using the notation $[X(z)]_{\downarrow 2}$ to denote the Z-transform of $x(2n)$, we can write this condition as

$$[D(z)\, D(-z)]_{\downarrow 2} \mid H_{02}(z).$$

Similarly, it can be shown that $\left(1 - z^{-1}\right)^{K_1} \mid H_{02}(z)$. Therefore,

$$H_{02}(z) = [D(z)\, D(-z)]_{\downarrow 2} \; \left(1 - z^{-1}\right)^{K_1} \; V(z)$$

where $V(z)$ is determined through spectral factorization according to (42).

Once we obtain $H_{02}(z)$ we have the first column of $\hat{H}(z)$. The remaining two columns of $\hat{H}(z)$ can be found using existing algorithms for paraunitary completion, for example those described in [26, 27] or [14]. Once the 3×3 paraunitary matrix $\hat{H}(z)$ is completely known, the 2×3 matrix $H(z)$ is obtained by deleting the last row of $\hat{H}(z)$.

Define $E_0(z)$ to be the first column of $H(z)$ (now known),

$$E_0(z) := \begin{bmatrix} H_{00}(z) \\ H_{01}(z) \\ H_{02}(z) \end{bmatrix}.$$

Then $E_0(z)$ is a 3×1 lossless system and as such it can be factored as follows [26].

$$E_0(z) = U_N(z) \cdot U_{N-1}(z) \, \cdots \, U_1(z) \cdot P \tag{46}$$

with

$$U_k(z) = I_3 - u_k \, u_k^t + u_k \, u_k^t \, z^{-1}$$

where u_k and P are column vectors of unit norm. The minimal number of factors N is the McMillan degree of the system $E_0(z)$. The McMillan degree also gives the minimum number of delay elements required to implement a system.

Once the factorization (46) is determined, using the algorithm described in [26, 27], a paraunitary matrix is obtained by replacing P with an orthogonal matrix Q ($Q^t \, Q = I_3$) the first column of which is P. The resulting paraunitary matrix will have the same McMillan degree as $E_0(z)$.

Note that $U_k(1) = I_3$. Then setting $z = 1$ in (46) gives

$$E_0(1) = U_N(1) \cdot U_{N-1}(1) \, \cdots \, U_1(1) \cdot P = P.$$

The column vector P is therefore uniquely determined by $H_{0i}(z)$. Note that $H_1(1) = H_2(1) = 0$ and hence $P_1(1) = P_2(1) = 0$; so from (36) we have $H_0(1) = \sqrt{2}$. Also note that $H_0(-1) = 0$. Therefore, from (38) we have

$$\begin{aligned} H_0(1) =& H_{00}(1) + H_{01}(1), \\ H_0(-1) =& H_{00}(1) - H_{01}(1). \end{aligned}$$

and in turn it follows that $H_{00}(1) = H_{01}(1) = 1/\sqrt{2}$. From (41) we have $H_{02}(1) = 0$. Hence, the column vector P is given by

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \tag{47}$$

Therefore, a 3×3 paraunitary matrix $\hat{H}(z)$, with $E_0(z)$ as the first column, is given by

$$\hat{H}(z) = U_N(z) \cdot U_{N-1}(z) \cdots U_1(z) \cdot Q$$

where Q is a 3×3 orthogonal matrix the first column of which is P in (47). In this case there is one degree of freedom in parameterizing Q. A simple parameterization of Q is given by

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_1) & -\sin(\alpha_1) \\ 0 & \sin(\alpha_1) & \cos(\alpha_1) \end{bmatrix}.$$

We will use the parameter α_1 to set the last coefficient of $h_2(n)$ to zero. It turns out that selecting α_1 in this way generally has the effect of making the last *two* coefficients of $h_2(n)$ equal to zero. (That is, $h_2(n)$ is two samples shorter than $h_0(n)$ and $h_1(n)$.) This effect was also observed in designing filters for the double-density DWT as discussed in [17]. It being two samples shorter than $h_0(n)$ and $h_1(n)$, let us shift $h_2(n)$ by two samples. The PR conditions will still be satisfied. A new set of solutions can be obtained by then applying an orthogonal rotation operation to $h_1(n)$ and $h_2(n-2)$ as follows,

$$\begin{bmatrix} h_1(n) \\ h_2(n) \end{bmatrix} \leftarrow \begin{bmatrix} \cos(\alpha_2) & -\sin(\alpha_2) \\ \sin(\alpha_2) & \cos(\alpha_2) \end{bmatrix} \begin{bmatrix} h_1(n) \\ h_2(n-2) \end{bmatrix}$$

where α_2 is selected so as to again set the last coefficient of the new $h_2(n)$ to zero. In our examples, we found that again, the last *two* coefficients of $h_2(n)$ turn out to be equal to zero. This procedure can be repeated until $h_2(n)$ is shorter than $h_0(n)$ and $h_1(n)$ by only one sample. Similar to what was observed in [17], this procedure has the effect of improving the shift relation between the two filters. Each time the shift and rotation is applied, the newly obtained filters $h_1(n)$ and $h_2(n)$ more nearly satisfy $h_1(n) = h_2(n-1)$ and the corresponding wavelets more nearly satisfy $\psi_{h,1}(t) = \psi_{h,2}(t-0.5)$. In addition, it does not increase the support of $h_0(n)$ or $h_1(n)$ and increases the support of $h_2(n)$ by only one sample. Although this procedure may not be optimal for obtaining wavelets satisfying (2) it is quite effective.

We note that we found that the filter bank completion procedure outlined above to be prone to numerical problems when used for the examples given in the next section. This is likely due to the high order zeros at $z = \pm 1$. However, by implementing the procedure in Maple and setting the arithmetic precision to 50 decimal digits the examples were computed to very high accuracy and this problem was entirely overcome. The filter bank completion algorithm of [14] may be more robust numerically, so it may offer another alternative.

6 Examples

With $K_0 = 4$, $K_1 = K_2 = 2$ and $L = 2$, the filters obtained by following the design procedure described above are tabulated in Table 2. The two scaling functions $\phi_h(t)$ and $\phi_g(t)$ and the four wavelet functions $\psi_{h,i}(t)$, $\psi_{g,i}(t)$, are shown in Figure 5. The integer translates of the four wavelets and their dyadic dilations form a (tight) frame. (A tight frame is one where the signal reconstruction can be performed with the transpose of the forward transform.) All of the wavelets have two vanishing moments. Note that $\psi_{h,1}(t) \approx \psi_{h,2}(t - 0.5)$ and $\psi_{g,1}(t) \approx \psi_{g,2}(t - 0.5)$ as illustrated in Figure 6. That is, the filters $\{h_0, h_1, h_2\}$ form a double-density DWT, and similarly $\{g_0, g_1, g_2\}$ also form one. In addition we have $\psi_{g,1}(t) \approx \mathcal{H}\{\psi_{h,1}(t)\}$ and $\psi_{g,2}(t) \approx \mathcal{H}\{\psi_{h,2}(t)\}$; accordingly, the spectrum of the complex wavelet $\psi_{h,i}(t) + j\,\psi_{g,i}(t)$ is approximately single sided. A complex double-density DWT is formed by taking $\psi_{h,i}(t)$ and $\psi_{g,i}(t)$ to represent the real and imaginary parts of a single complex wavelet. Figure 6 also illustrates the envelop function $\sqrt{\psi_{h,i}(t)^2 + \psi_{g,i}(t)^2} = |\psi_{h,i}(t) + j\,\psi_{g,i}(t)|$, the magnitude of the complex wavelet. As expected of a bandpass function whose spectrum is single sided, the envelop has a bell-shape.

When we take $K_0 = 6$, $K_1 = K_2 = 3$ and $L = 3$, we obtain the filters tabulated in Table 3. The wavelets are illustrated in Figure 7 and 8. In this example, the wavelets have 3 vanishing moments. Compared with the previous example, they more nearly satisfy $\psi_{g,i}(t) = \mathcal{H}\{\psi_{h,i}(t)\}$, $\psi_{h,1}(t) = \psi_{h,2}(t - 0.5)$ and $\psi_{g,1}(t) = \psi_{g,2}(t - 0.5)$ as illustrated in Figure 8.

n	$h_0(n)$	$h_1(n)$	$h_2(n)$
0	0.06911582051268	0.00007342378571	0.00016216892384
1	0.35966127039020	0.00038207883303	0.00084388611367
2	0.66578510237730	-0.00598664482533	-0.01366169688553
3	0.46591894334945	-0.03433855126018	-0.07812787935415
4	-0.01910143987094	-0.05544284199193	-0.08404354646603
5	-0.13775229565904	0.00187143276342	0.22307058313047
6	-0.00879228135716	0.13862717458070	0.39450869604392
7	0.01947949838578	0.33211688782429	-0.65664993171756
8	0.00009957952467	-0.56616644388191	0.21389772021137
9	-0.00020063527985	0.18886348417221	0
n	$g_0(n)$	$g_1(n)$	$g_2(n)$
0	0.01382316410254	0.00036711892855	0.00081084461919
1	0.18251756689833	0.00484734559355	0.01070618752183
2	0.55379561515138	0.01295727262146	0.02642247548195
3	0.64032052013227	-0.00610823096768	-0.04248472459154
4	0.20240253780720	-0.06568401496723	-0.20956025899924
5	-0.13270357516508	-0.09685196235772	-0.00551846607616
6	-0.07143784469173	-0.02112084546363	0.65041073668340
7	0.01797544572026	0.54923548323017	-0.47356633868171
8	0.00852330881716	-0.41541486345192	0.04277954404227
9	-0.00100317639923	0.03777269683444	0

Table 2: The coefficients for Example 1. $K_0 = 4$, $K_1 = K_2 = 2$, $L = 2$.

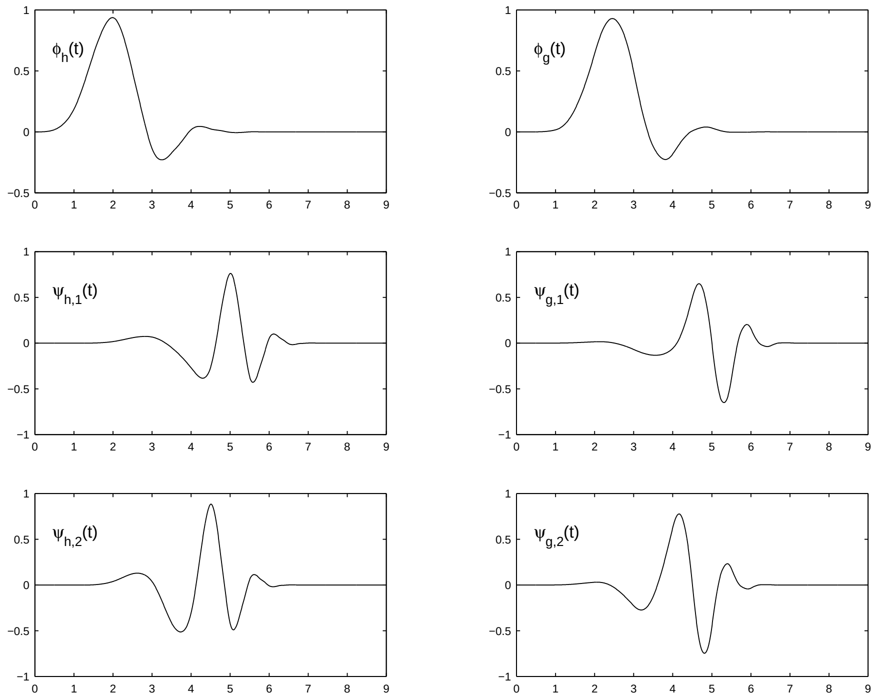

Figure 5: Example 1, $K_0 = 4$, $K_1 = K_2 = 2$, $L = 2$. The filter coefficients are tabulated in Table 2.

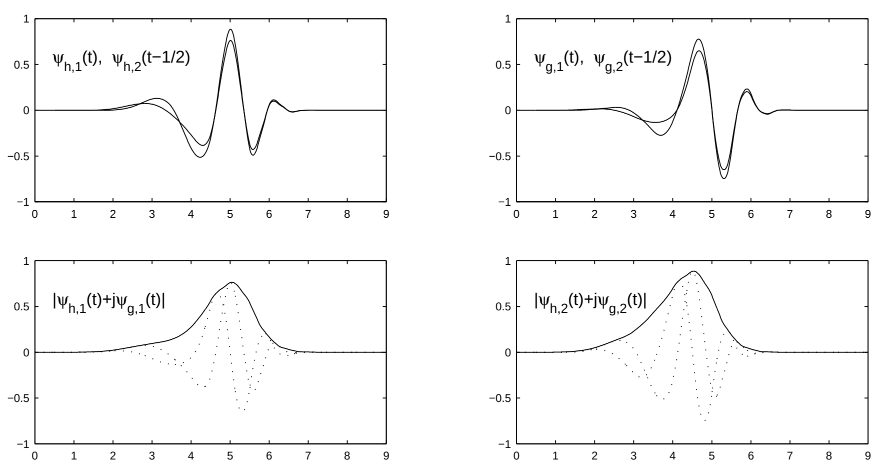

Figure 6: Example 1. Illustration of the shift property and the envelope for the wavelets illustrated in Figure 5.

n	$h_0(n)$	$h_1(n)$	$h_2(n)$
0	0.01167515006693	0.00000028033779	0.00000096316386
1	0.11210453438921	0.00000269179728	0.00000924827821
2	0.39020359888747	-0.00009458242812	-0.00032856572032
3	0.63766002212216	-0.00098283172247	-0.00341136920868
4	0.45159271169876	-0.00322600807907	-0.00984858347245
5	-0.01779052717512	-0.00339847237922	0.00114352817725
6	-0.18995098895800	0.00534784540227	0.05358462859672
7	-0.03633171373566	0.02694106074718	0.07100034042574
8	0.05116380416058	0.04999293344040	-0.07326560617712
9	0.01309797742830	-0.00764246645306	-0.23356729557382
10	-0.00814108745277	-0.21155330115108	-0.04788025853496
11	-0.00163786108717	-0.13672353553244	0.58084573581233
12	0.00056506737003	0.61809721274879	-0.40145448519473
13	0.00000434924483	-0.39817251892698	0.06317171942796
14	-0.00000147458644	0.06141169219871	0

n	$g_0(n)$	$g_1(n)$	$g_2(n)$
0	0.00166787858099	0.00000196236451	0.00000674214701
1	0.04270099078000	0.00005024041316	0.00017261229964
2	0.23192413512891	0.00023596311036	0.00078545980855
3	0.54594099119669	-0.00030264225957	-0.00168611304984
4	0.60903833681124	-0.00443438247636	-0.01814247165657
5	0.21459366376196	-0.01230171877677	-0.03508479826925
6	-0.16295875588933	-0.01563309030618	0.01806298323729
7	-0.12839582439678	0.00449550768918	0.13569634310037
8	0.03096765367958	0.07816842450995	0.09808771813111
9	0.03738202154037	0.13192700814758	-0.19634137757248
10	-0.00385258125359	-0.12443537362415	-0.37624919671465
11	-0.00531066009030	-0.44659309709664	0.56741070948369
12	0.00033043623380	0.57729947000603	-0.20174314229173
13	0.00019559839462	-0.19725137058663	0.00902453134685
14	-0.00001032210505	0.00877309888553	0

Table 3: The coefficients for Example 2. $K_0 = 6$, $K_1 = K_2 = 3$, $L = 3$.

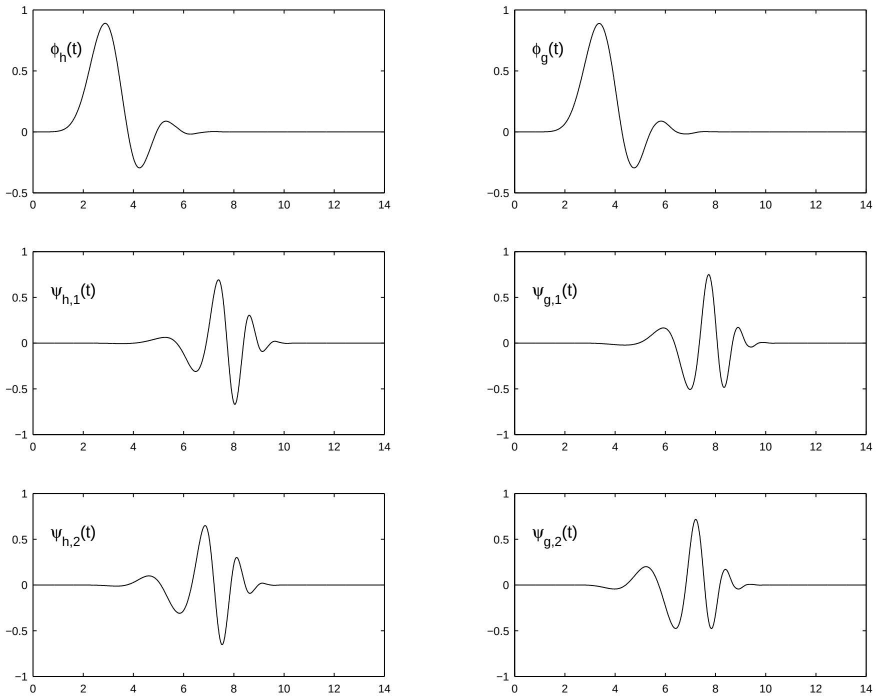

Figure 7: Example 2, $K_0 = 6$, $K_1 = K_2 = 3$, $L = 3$. The filter coefficients are tabulated in Table 3.

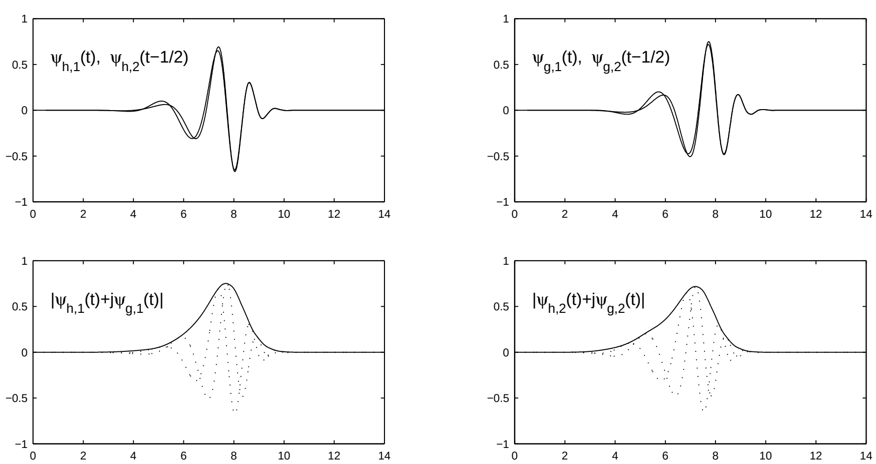

Figure 8: Example 2. Illustration of the shift property and the envelope for the wavelets illustrated in Figure 7.

7 Conclusion

This paper has introduced the double-density dual-tree DWT, a discrete wavelet transform that has properties of both the double-density DWT [17] and the dual-tree DWT [10]. The filter bank structure corresponding to this DWT consists of a pair of iterated oversampled filter banks operating in parallel. Each iterated filter bank is of the form described in [17] and is redundant by a factor of 2. The total redundancy is therefore a factor of 4. The 4 wavelets are designed so that they form two approximate Hilbert transform pairs, and so that the integer translates of one pair fall midway between the integer translates of the other pair. The paper has also developed a design procedure to obtain FIR filters which satisfy the numerous constraints imposed. This design procedure, which draws from the methods described in [3, 17] and [18, 19], employs a fractional-delay all-pass filter, spectral factorization, and filter bank completion.

A Condition on $H_{02}(z)$

By expanding in terms of polyphase components, it can be shown that

$$H_0(z)\, H_0(1/z) + H_0(-z)\, H_0(-1/z) = 2\left[H_{00}(z^2)\, H_{00}(1/z^2) + H_{01}(z^2)\, H_{01}(1/z^2)\right].$$

Therefore, the equation (41) can be rewritten as

$$\begin{aligned} H_{02}(z^2)\, H_{02}(1/z^2) &= 1 - \frac{1}{2}\left[H_0(z)\, H_0(1/z) + H_0(-z)\, H_0(-1/z)\right] && (48)\\ &= 1 - \frac{1}{2} P_0(z) - \frac{1}{2} P_0(-z). && (49) \end{aligned}$$

As $P_0(z) + P_1(z) + P_2(z) = 2$ from (36), we can write $1 - \frac{1}{2}\, P_0(-z) = \frac{1}{2}\, P_1(-z) + \frac{1}{2}\, P_2(-z)$,

$$H_{02}(z^2)\, H_{02}(1/z^2) = -\frac{1}{2}\, P_0(z) + \frac{1}{2}\, P_1(-z) + \frac{1}{2}\, P_2(-z).$$

From (32) and (37) we can write this as

$$H_{02}(z^2)\, H_{02}(1/z^2) = \frac{1}{2}\left[-D(z)\, D(1/z)\, (z + 2 + z^{-1})^{K_0}\, R_0(z) + D(z)\, D(1/z)\, (z + 2 + z^{-1})^{K_1}\, R_{12}(-z)\right]$$

or

$$H_{02}(z^2)\, H_{02}(1/z^2) = \frac{1}{2}\, D(z)\, D(1/z)\, (z + 2 + z^{-1})^{K_1}\left[R_z\, (z + 2 + z^{-1})^{K_0 - K_1} - R_{12}(-z)\right]$$

That is,

$$D(z)\, D(1/z)\, (z + 2 + z^{-1})^{K_1} \mid H_{02}(z^2)\, H_{02}(1/z^2).$$

As $H_{02}(z^2)\,H_{02}(1/z^2)$ is an even function of z, we have in fact, the condition

$$D(z)\,D(1/z)\,D(-z)\,D(-1/z)\,(z^2+2+z^{-2})^{K_1} \mid H_{02}(z^2)\,H_{02}(1/z^2).$$

To derive (44) note from (40) that

$$H_{00}(z)\,H_{10}(1/z)+H_{01}(z)\,H_{11}(1/z)+H_{02}(z)\,H_{12}(1/z)=0$$

Also note (by expanding in terms of polyphase components) that

$$H_0(z)\,H_1(1/z)+H_0(-z)\,H_1(-1/z)=2\left[H_{00}(z^2)\,H_{10}(1/z^2)+H_{01}(z^2)\,H_{11}(1/z^2)\right].$$

Therefore,

$$H_{02}(z^2)\,H_{12}(1/z^2)=-\frac{1}{2}\,\left[H_0(z)\,H_1(1/z)+H_0(-z)\,H_1(-1/z)\right].$$

From (16) and (17) we then have that

$$D(z)\,D(-z) \mid H_{02}(z^2)\,H_{12}(1/z^2).$$

The condition (45) is derived similarly.

References

[1] P. Abry. *Ondelettes et Turbulences.* Diderot, Paris, 1997.

[2] P. Abry and P. Flandrin. Multiresolution transient detection. In *Proc. of the IEEE-SP Int. Symp. Time-Frequency and Time-Scale Analysis*, pages 225–228, Philadelphia, October 1994.

[3] C. Chui and W. He. Compactly supported tight frames associated with refinable functions. *Applied and Computational Harmonic Analysis*, 8(3):293–319, May 2000.

[4] C. K. Chui, W. He, and J. Stöckler. Compactly supported tight and sibling frames with maximum vanishing moments. 2001. Preprint.

[5] I. Daubechies, B. Han, A. Ron, and Z. Shen. Framelets: MRA-based constructions of wavelet frames. 2001. Preprint.

[6] F. Fernandes, R. van Spaendonck, and C. S. Burrus. A new directional, low-redundancy complex-wavelet transform. In *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing (ICASSP)*, May 2001.

[7] W. T. Freeman and E. H. Adelson. The design and use of steerable filters. *IEEE Trans. Patt. Anal. Mach. Intell.*, 13(9):891–906, September 1991.

[8] N. G. Kingsbury. The dual-tree complex wavelet transform: a new technique for shift invariance and directional filters. In *Proceedings of the Eighth IEEE DSP Workshop*, Utah, August 9-12, 1998.

[9] N. G. Kingsbury. Image processing with complex wavelets. *Phil. Trans. Royal Society London A*, September 1999.

[10] N. G. Kingsbury. Complex wavelets for shift invariant analysis and filtering of signals. *Applied and Computational Harmonic Analysis*, 10(3):234–253, May 2001.

[11] T. I. Laakso, V. Välimäki, M. Karjalainen, and U. K. Laine. Splitting the unit delay. *IEEE Signal Processing Magazine*, 13(1):30–60, January 1996.

[12] A. Petukhov. Explicit construction of framelets. Research report 00:03, Industrial Mathematics Institute, University of South Carolina, 2000. `http://www.math.sc.edu/~imip/`.

[13] A. Petukhov. Symmetric framelets. Research report 00:15, Industrial Mathematics Institute, University of South Carolina, 2000. `http://www.math.sc.edu/~imip/`.

[14] C. Popeea, B. Dumitrescu, and B. Jora. An efficient algorithm for filter bank completion. In *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing (ICASSP)*, May 2001.

[15] A. Ron and Z. Shen. Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator. *Journal Funct. Anal.*, 148:408–447, 1997.

[16] A. Ron and Z. Shen. Construction of compactly supported affine frames in $L_2(\mathbb{R}^d)$. In K. S. Lau, editor, *Advances in Wavelets*. Springer Verlag, 1998.

[17] I. W. Selesnick. The double density DWT. In A. Petrosian and F. G. Meyer, editors, *Wavelets in Signal and Image Analysis: From Theory to Practice*. Kluwer. To appear.

[18] I. W. Selesnick. The design of Hilbert transform pairs of wavelet bases. *IEEE Trans. on Signal Processing*, 2001. In review.

[19] I. W. Selesnick. The design of Hilbert transform pairs of wavelet bases via the flat delay filter. In *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing (ICASSP)*, May 2001.

[20] I. W. Selesnick. Hilbert transform pairs of wavelet bases. *IEEE Signal Processing Letters*, 8(6):170–173, June 2001.

[21] I. W. Selesnick. Smooth wavelet tight frames with zero moments. *Applied and Computational Harmonic Analysis*, 10(2):163–181, March 2001.

[22] I. W. Selesnick and L. Sendur. Smooth wavelet tight frames with denoising applications. In *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing (ICASSP)*, June 2000.

[23] E. P. Simoncelli and W. T. Freeman. The steerable pyramid: A flexible architecture for multi-scale derivative computation. In *Proc. IEEE Int. Conf. Image Processing*, Washington, DC, October 1995.

[24] E. P. Simoncelli, W. T. Freeman, E. H. Adelson, and D. J. Heeger. Shiftable multi-scale transforms. *IEEE Trans. Inform. Theory*, 38(2):587–607, March 1992.

[25] J. P. Thiran. Recursive digital filters with maximally flat group delay. *IEEE Trans. on Circuit Theory*, 18(6):659–664, November 1971.

[26] P. P. Vaidyanathan. *Multirate Systems and Filter Banks*. Prentice Hall, 1993.

[27] P. P. Vaidyanathan, T. Q. Nguyen, Z. Doğanata, and T. Saramäki. Improved technique for design of perfect reconstruction FIR QMF banks with lossless polyphase matrices. *IEEE Trans. on Acoust., Speech, Signal Proc.*, 37(7):1042–1056, July 1989.