DETAILED MODELING OF EDDY CURRENT EFFECTS FOR TRANSFORMER TRANSIENTS

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Abstract. A detailed time domain model for the eddy current losses in the windings of a transformer is presented. The basic elements for the derivation of the model are the turns which may be combined into sections. The model is expressed as an R-matrix. Its diagonal elements were fitted using a series Foster circuit while for the off-diagonal elements we used the basis functions of the Foster circuits to derive the time domain model in the form of state equations. For validation, the losses computed with the detailed model are compared with those obtained by considering full windings. The frequency response of the transformer model is compared with test results. Simulations are presented for illustration and further validation.

Keywords: Eddy currents, Transformer modeling, Electromagnetic transients, State equation approximation, Transfer function matrices.

INTRODUCTION

This paper is in continuation of our efforts in the development of a transformer model for the calculation of transients. Its objective is to contribute to the understanding and modeling of eddy current effects in the turns of the transformer. It is not, however, within its scope to deal with procedures for deriving circuits that give the right resonance locations. Rather, our main concern here is to obtain the value of the winding resistance over a specified frequency range (including at resonances). The complete model, which includes the losses in the windings as well as in the iron core, was presented in a separate paper [1].

Previously, we have calculated the leakage inductances and capacitances for the turns of a transformer [2] and have derived a transformer model, based on elementary parameters, for the calculation of transients [3]. In reference [4] we gave a time domain model in the form of a Foster circuit for the calculation of the losses for full windings. The validity of the model presented in reference [4] is however restricted to a relatively low frequency range since it does not permit the inclusion of the capacitive effects between turns or sections of a winding. The object of the present paper is the derivation of a more general time domain model in the form of state equations for the detailed representation, on a turn-to-turn basis, of eddy current effects in the windings of transformers.

There exists a detailed transformer model which includes the losses in the windings; see references [5] and [6]. This model is derived from the calculation of self and mutual impedances in transformers. Quantities related to leakage impedances can be, however, inaccurate being differences between nearly equal numbers (self and mutual impedances). To get accurate leakage related quantities and losses, the authors of [5] and [6] rely on tests to compute some adjustment parameters. Another detailed model derived from classical transformer theory was presented in reference [7]. This model is capable of handling the frequency dependent losses in the transformer windings, however, it also uses self and mutual inductances with the inconveniences noted above. In other instances [8]-[12], the problem of ill-conditioning has been adequately solved in transient simulations by subtracting a common flux in the calculation of self and mutual inductances; see reference [8] for the details. The same methodology can be applied in the models of references [5] to [7]. In fact, subtracting a large common quantity from the self and mutual inductances is equivalent to the direct use of leakage impedances we have adopted for the modeling of transformers.

The steps for the derivation of the state equations are the following:

a) An R-matrix is computed from the loss equations of individual turns. This matrix results from the sum of the quadratic forms for the losses in the conductors, produced by the currents in all conductors. The matrix is first modified to take into account that the sum of the currents inside of a window is zero and later a reduced order $R''$ matrix is obtained by lumping turns in series to form sections.

b) The diagonal elements of $R''$ are fitted with series Foster circuits (using the real-only approach described in reference [4]).

c) The fitting for the off-diagonal elements is obtained as a linear combination of the same basis functions which constitute the diagonal elements (resulting from the poles of the Foster circuits).

d) The state equations for the representation of eddy current effects in the transformer windings are formulated.

The fitting method presented in this paper can be applied to any smooth transfer function matrix from other fields of study and it is thus not limited to the modeling of the losses in transformer windings.

EDDY CURRENT LOSSES IN INDIVIDUAL TURNS

The impedance (in $\Omega/m$) of a conductor with circular cross section is given by the well-known expression [13]:

$$Z = \frac{\alpha}{2 \pi \sigma} \frac{J_0(\alpha a)}{J_0(\alpha a)}$$

(1)

where

$$\alpha = \sqrt{i} \left( \mu \omega \right)$$

$J_n$ = modified Bessel functions of the first kind and order $n$

$\sigma$ = conductor radius

$\omega$ = angular frequency

$\mu$ = permeability

The real part of equation (1) gives the skin effect losses which represent the most significant part of the total losses in an individual turn of a winding at low frequencies. Note that in the case of a conductor of rectangular cross section we are still using a circular cross section to approximate the skin effect losses. We select the radius of the circle such that the two areas are the same.

The complex power (in VA/m) of a rectangular conductor (that becomes relevant at high frequencies) when the magnetic field at its surface is specified (as in the case of laminations; see reference [13]) is given by

$$S_c = \frac{2W}{\sigma d} \left( \frac{H^2}{\kappa_2} \right)$$

(2)

where

$\kappa_2 = \alpha d$

$H$ = magnetic field strength (r.m.s. value)

$d$ = half of the conductor’s thickness

$W$ = conductor height (= 2d for a square conductor)

Equation (2) will be used to compute the losses in a conductor due to the currents in all other conductors. These are commonly called proximity effect losses and, as we will later illustrate, at high frequencies they are much larger than the losses due to skin effect.

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THE R-MATRIX

In this section we will derive an R-matrix which is useful for computing the voltage drop related to the eddy current losses in the windings. The off-diagonal elements of the matrix represent the voltage drop in a turn produced by the currents in other turns while the diagonal elements include also skin effect losses. At the final stage, a set of differential equations, derived from the frequency dependent R-matrix, will be used instead of the resistance matrix in the voltage equation of the transformer windings. The R-matrix will be obtained from the total losses in the windings and therefore it will represent accurately only the total effect of the eddy currents. Information regarding the exact location of the losses cannot be extracted from the R-matrix but can be obtained using the individual matrices from which the matrix R has been built. Later in this section, we will compare the total resistance with the one obtained using the full winding model. We will also show that as the frequency increases the losses due to proximity effect increase faster than the losses due to skin effect. Finally, we will modify the R-matrix to account for the fact that the total current in a window is zero (for an ideal iron core), and we will lump the turns to form sections in order to reduce the matrix to a manageable size.

Derivation of the R-Matrix

The losses expression (2) is a function of the square of the total magnetic field \( H \) (r.m.s. value). We can estimate the magnetic field at any point in the window with the use of the image method developed in reference [2]. The two components of the magnetic field of a circular filament carrying current at any point are given [14] by

\[
H_x = \frac{I}{2\pi} \frac{z}{r_0 \sqrt{(r_x + r_0)^2 + z^2}} \left[ K(k) + \frac{r_x^2 + r_0^2 + z^2}{(r_x - r_0)^2 + z^2} E(k) \right] (3)
\]

and

\[
H_y = \frac{I}{2\pi} \frac{1}{\sqrt{(r_x + r_0)^2 + z^2}} \left[ K(k) + \frac{r_x^2 - r_0^2 - z^2}{(r_x - r_0)^2 + z^2} E(k) \right] (4)
\]

where

\[
k = \sqrt{\frac{4 r_x r_0}{z^2 + (r_x + r_0)^2}}
\]

\( I = \) current on the filament

\( E(k) = \) elliptic integral of second kind and argument \( k \)

\( K(k) = \) elliptic integral of first kind and argument \( k \)

\( r_0 = \) radius of the observation point

\( r_x = \) radius of the excited filament

\( z = \) vertical separation between the filament and the observation point

The radial and axial components of the magnetic field produced at the center of a conductor \( m \) by the currents in all other conductors are given by

\[
H_x = i^T f_i (6a)
\]

\[
H_y = i^T f_i (6b)
\]

where \( i \) is the complex vector of r.m.s. current phasors in all conductors, except \( m \), and \( f_i, f_j \) are real vectors of coefficients \( f_{ij}, f_{ji} \) functions of elliptic integrals; see Appendix A for details. Note that, for notational simplicity, the identifier \( m \) is not yet explicitly shown in the equations.

Consequently,

\[
|H|^2 = H_x H_x^* + H_y H_y^* = i^T (f_i f_j^* + f_j f_i^*) i^* = \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{ij} I_i I_j (7a)
\]

or in matrix form,

\[
|H|^2 = i^T \beta i^* = \begin{bmatrix} I_1 & \beta_{12} & \cdots & \beta_{1N} \\ I_2 & \beta_{21} & \cdots & \beta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ I_N & \beta_{N1} & \cdots & \beta_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}^* (7b)
\]

where

\[
\beta_{ij} = f_{ij} f_{ij}^* + f_{ji} f_{ji}^*
\]

When using the quadratic form to compute the field (and later the losses) at the center of a conductor, we shall consider independently the losses due to the current in that conductor. Therefore, at this stage, the quadratic form has both row \( m \) and column \( m \) equal to zero.

Substituting equation (7a) into equation (2) we obtain the losses in the conductor \( m \) due to the currents in all other conductors as

\[
P_m = F_m(u) \sum_{i=1}^{N} \sum_{j=1, j \neq m}^{N} \beta_{ij} I_i I_j (9)
\]

where

\[
F_m(u) = \text{Re} \left\{ \frac{2 \pi}{\sigma d} \xi \text{tanh}(\xi) \right\} (10)
\]

The skin effect losses due to the current in conductor \( m \) can be computed from equation (1) as

\[
P_m = R_m I_m^2 (11)
\]

Equations (9) and (11) combine to a quadratic form for the total losses in conductor \( m \)

\[
P_m = P_m' + P_m'' = \sum_{i=1}^{N} \sum_{j=1, j \neq m}^{N} R_{ij} I_i I_j (12)
\]

where

\[
R_{ij} = \begin{cases} \text{Re}(Z) & i,j \neq m \\ 0 & \text{otherwise} \end{cases} (13)
\]

The total losses in the winding will be given by the following quadratic form

\[
P_{total} = \sum_{m=1}^{N} P_m = \sum_{i=1}^{N} \sum_{j=1}^{N} (R_{ij} I_i I_j) (14)
\]

which can also be written as a sum of matrices. Equation (14) defines the R-matrix with the general element given by

\[
R_{ij} = \sum_{m=1}^{N} R_{ij} (15a)
\]

or

\[
R = \sum_{m=1}^{N} R_m (15b)
\]

Local losses (for finding possible hot spots) can be calculated from the matrices \( R_m \) of equation (15b), working backwards, once the full solution has been obtained.

Comparison with Full Windings

In reference [4] we have computed the total resistance for the windings of a transformer when the windings are considered as complete entities using the well-known equation which assumes that the magnetic field is axial [15]-[19]. In reference [19], Figure 2-9, the results of an indirect experiment for the estimation of the resistance have been compared with theoretical results. The tests have shown that the high frequency resistance, predicted with the full winding approach, is 25 to 30% higher than the measured resistance. These measurements are difficult to perform since the power factor is very less, than 5% and very often less than 1%; see reference [18] which presents a detailed discussion of measurement alternatives and procedures.

In Figure 1, we show the comparison of:

a) The resistance obtained by considering full windings.

b) The resistance obtained with the R-matrix and short circuited square conductors.

We can see that, at low frequencies, the turn-to-turn approach matches perfectly the prediction with full windings. At high frequencies, the turn-to-turn approach predicts smaller losses than the full winding method, by up to 25%. This bring the new computed total resistance close to the above mentioned test results.
The Augmented $R'$ - Matrix

In the model for the transformer described in reference [3] we have used two fictitious turns to interface the iron-core equations with the windings equations. The fictitious turn $\alpha$ is located on the leg (at the middle, at the surface) while the fictitious turn $\beta$ is located on the yoke. The turns $\alpha$ and $\beta$, which carry the magnetizing currents, are included in the $R$ matrix (as if they were two additional real turns). Thus, we have an augmented $R'$ - matrix of order $N+2$ which is calculated by adding only $N$ quadratic forms as in equation (14), since the resistivity of the fictitious turns is considered to be zero. There are no losses produced in turns $\alpha$ and $\beta$, but the currents in these two turns will induce eddy currents in all other turns. The subscripts $i$ and $j$ in equation (14) will take $N+2$ values while the superscript $\alpha$ will go to $N$ only.

We also have to consider the fact that the total current inside of a window is zero. This is the base of our magnetic model for transformers; see reference [2]. We follow the same procedure as in the case of the loop inductance matrix $L'$ which consists of:

a) Taking turn $\alpha$ as the reference ($v_i - v_{\alpha}$ for $i = 1, 2, \ldots, N, \beta$)

b) Using $\sum_{s=1}^{N+2} i_s = 0$

The general term of the loop $R'$ - matrix is a function of the augmented $R$ - matrix and it is given by

$$R'_{ij} = R_{ij} - R_{i\alpha} - R_{\alpha j} + R_{\alpha \alpha}$$

(16)

We have taken turn $\alpha$ as the reference turn because its flux is the common flux for all turns. The order of the loop $R'$ - matrix is $N+1$ which is the number of loops. This matrix reflects the fact that we have only $N+1$ independent currents in the model for the transformer.

Reduction of the $R'$ - Matrix

The loop $R'$ - matrix has the very high order ($N+1$) but it can be reduced to sections. The reduction process consists of lumping a number of turns in series to form sections. The new reduced matrix, $R''$, is obtained by adding all the elements in the $R'$ - matrix which correspond to the section being formed. In reference [3] we have shown all the details and an example for the analogous case of inductance matrices.

TIME DOMAIN APPROXIMATION OF THE $R''$ - MATRIX

In this section we develop a state equations model for the voltage drop caused by the eddy currents in the windings of a transformer. We will treat independently the diagonal and the off-diagonal elements.

Fitting of Diagonal Elements

For the diagonal elements we use the experience gained in reference [4]. We fit a series Foster circuit to the frequency dependent resistance which is given at discrete frequencies. Figure 3 shows a series Foster circuit of order $n$.

For a series Foster of order $n$ we need $2n + 1$ evaluations of the $R''$ - matrix; see Figure 3. The first frequency should be zero, for obtaining $r_n$, but we use 1 Hz to avoid numerical problems. The other $2n$ frequencies can be simply selected in decades starting from 1 kHz, i.e., 1, 10, 100 kHz, etc., to assure convergence.
We obtain reasonable accuracy over the whole frequency range (up to 1 MHz) when we use a model of order 2 (two blocks plus the d.c. resistance). More accurate responses were obtained using the following frequencies: 1 Hz, 7 kHz, 13 kHz, 130 kHz, and 1 MHz, obtained by trial and error. The initial values for resistances and inductances must be very small since some of the diagonal elements of \(R''\) are very small numbers. We used \(10^{-12}\) for all initial resistances and inductances and convergence was obtained in 10 iterations in all cases. In Figure 4 we show a typical frequency variation curve. This plot corresponds to the 236 turn transformer described in Appendix 2 of reference [2]. We present the frequency variation of the first diagonal element of the \(R''\) matrix once the 236 turns have been reduced to 4 sections of 59 turns each. We can see that the response of a series Foster circuit of order 2 represents fairly accurately the original function.

![Figure 4. Frequency variation of the diagonal element \(R''_{11}\) (of Figure 3 is not included in the plotting)](image)

**Fitting of Off-Diagonal Elements**

It is assumed that all elements in each column of a transfer function matrix can be represented by a linear combination of a small number of simple fractions, each having one pole. These will henceforth be called basis functions. The rationale for this assumption is that each input, say current \(I_1\), excites a number of modes (or basis functions) and a linear combination of these is expected to be observed as components in any of the outputs, for instance in voltage \(V_j\).

Once a diagonal element has been fitted, the off-diagonal elements in the respective column can be approximated by using a linear combination of the basis functions of the diagonal element (blocks of the series Foster circuit). We achieve this by expressing the total voltage drop in a turn (section) as a linear function of the state variables. The state variables are the currents in the inductances of the series Foster circuit fitted for the diagonal element of the \(R''\) matrix. The voltage drop in the \(j\)th block of the \(j\)th Foster circuit (i.e. the circuit fitted for the \(j\)th diagonal element) is given by

\[
v_{j\ell} = \sum_{k=1}^{n} \rho_{j,k} v_{k} + \rho_{j,0} i_j
\]

where

- \(r_{j,k}\) = resistance of the \(k\)th block of the \(j\)th Foster circuit
- \(L_{j,k}\) = inductance of the \(k\)th block of the \(j\)th Foster circuit
- \(i_j\) = total current in turn \(j\) in the \(j\)th Foster circuit
- \(v_{j,k}\) = current in the \(k\)th inductance of the \(j\)th Foster circuit
- \(\rho_{j,k}\) = mutual resistance of the \(j\)th Foster circuit

The total number of state variables is the number of turns times the order of the Foster circuits \((N \times n)\). From relation (17) we can obtain \(N \times n\) state equations of the form

\[
\frac{dv_{j,k}}{dt} = r_{j,k} s_{j,k} + r_{j,0} s_{j,k} i_j
\]

The voltage drop in turn \(j\), due only to the current in turn \(j\) itself (i.e. the combination of current at the terminals of the Foster circuit), is

\[
v_{j,j} = \sum_{k=1}^{n} \rho_{j,k} v_{k} + \rho_{j,0} i_j
\]

and can be expressed as a function of the state variables by substituting (17) into (19)

\[
v_{j,j} = \sum_{k=1}^{n} \rho_{j,k} v_{k} + \rho_{j,0} i_j
\]
approximations. We can see that the response of the approximations is remarkably good considering that we used a model of low order (only 3 frequencies).

![Figure 5. Frequency variation of the fitting of the off-diagonal elements $R'''_{12}$ and $R'''_{21}$](image)

**State Equation Form of the Model**

The equations derived above can be written in a complete state equation form. From equation (21) we can calculate the voltage drop in turn $i$ due to the current in turn $j$ once the $\rho_{ij}$ coefficients are known. Consequently, we can get the voltage drop in turn $i$ produced by the currents in all other turns. Thus, the total voltage drop in turn $i$ is

$$v_i = \sum_{j \neq i}^{N} \rho_{ij} v_j + v_i$$

(30)

where $v_i$ can be calculated with equation (20). Substituting equations (20) (for turn $i$) and equation (21) into equation (30) we get

$$v_i = \sum_{j \neq i}^{N} (\rho_{ij} x_{j1} + \rho_{ij} x_{j2} l_j) - \sum_{k=0}^{N} x_{ik} l_i + (\sum_{k=0}^{N} x_{ik}) l_i$$

(31)

Equations (18) and (31) can be applied to all turns and a set of state equations of the general form

$$\dot{x} = Ax + Bu$$

(32)

$$v_i = C x + D u_i$$

(33)

can be written. The detailed definition of the matrices is given in Appendix B.

**FREQUENCY RESPONSE**

The voltage equation for the windings on one limb of a transformer is given by equation (14) of reference [3] as:

$$v_i = w_a E_a + R_a l_i + L''_a \frac{d}{dt} l_i$$

(34)

The frequency response of the transformer (including the eddy current losses) is computed by substituting the $R_a l_i$ drop in equation (34) by $v_i$ obtained from the state equations (32) and (33) and setting $d/dt = j \omega$. We get the following voltage equation for phasors:

$$v_a = w_a E_a + v_i + j \omega L''_a l_i$$

(35)

The voltage drop in the windings can be expressed from equations (32) and (33) (see Appendix B) as

$$v_i = z_a l_i$$

(36)

where

$$z_a = C [j \omega I - A]^{-1} B + D$$

(37)

Substituting equation (36) into equation (35) yields

$$v_a = w_a E_a + Z z_a l_i$$

(38)

where $Z = Z_a + j \omega L''_a$ is the total impedance and includes the effect of the eddy current losses in the transformer windings.

The frequency domain transformer model given by equation (38) has to be interconnected with the iron-core model and the external system for the calculation of the total frequency response. The necessary steps are identical to those followed in reference [3] for the calculation of transients.

**TRANSIENT MODEL**

The transient model is obtained from equation (34) by calculating the voltage drop $v_i$ in the windings using the state equations (32) and (33). In the model presented in reference [3] the iron-core and winding equations are decoupled and solved iteratively. The state equations representing the eddy current losses in the windings (equations (32) and (33)) can also be considered as being decoupled and can be included in the same iteration cycle; see Figure 6.

![Figure 6. Decoupled transformer model](image)

**TEST RESULTS**

**Frequency Domain**

The frequency response in short circuit of three transformers was available: a small 2 kVA laboratory transformer, a 75 kVA distribution transformer, and a 93 MVA power transformer; see Figures 7, 8, and 9. The short circuit tests, needed to reflect eddy current losses in the windings (as opposed to iron losses), show resonances at significantly higher frequencies than open circuit or load tests, often reported in the literature; see, for example, reference [10]. This is due to the fact that in short circuit it is the leakage rather than the magnetizing inductance that dominantly affects the lowest resonance frequency. Reference [20] presents a more detailed discussion based on measurements.

In Figure 7 we present the measured and simulated variation of resistance with frequency of the 2 kVA transformer described in Appendix 2 of reference [2]. We can see that the overall response is quite accurate.

![Figure 7. Frequency variation of input resistance for the 2 kVA transformer](image)
Figure 8 compares the resistance obtained with the model against tests for the 75 kVA transformer of reference [20]. Since resistances are not easily measured due to the low power factor, only those values that were measured with reasonable accuracy are shown (with a small circle) in Figure 8. In reference [18] different measurement techniques are discussed. The simulated responses shown in Figure 8 have been obtained with an approximate transformer design. The effect of varying the layout of the windings is shown in the figure. In curve (a) no attempt was made to best fit the measurement data. Layouts (b) and (c) fit better the measurements. In the figure the d.c. resistance is shown as reference. It is interesting to note that in the low frequency range a winding with more layers has smaller resistance, while at high frequencies the opposite is true. Therefore, a design with a larger number of layers is better conditioned for both low frequency (smaller eddy losses in steady state) and high frequency (larger damping for transients) performance.

![Graph showing frequency variation of input resistance for the 75 kVA transformer.](image)

Figure 8. Frequency variation of input resistance for the 75 kVA transformer

a) 2 layers in primary and 8 layers in secondary
b) 4 layers in primary and 16 layers in secondary
c) 8 layers in primary and 32 layers in secondary

In Figure 8 we can see that there are two small resonance peaks that are not predicted with the approximate transformer design we have used, when we vary the number of layers in the windings. An improved fit was obtained when the capacitance to tank was included in the model. In Figure 9 we show the results of inserting the capacitance to the tank in three points of the external winding. Thus the fitting has been improved to some extent. There still exist other losses not considered in the model. Specifically, the model does not include the losses produced in the dielectric and by stray fluxes. There are also construction details that were not considered, e.g., grading rings, tap changers, bushings, etc. All these, combined with the effects of discretization, reduce the accuracy and validity of the model particularly at the very high frequencies displayed in Figure 8 and 9.

Test and simulation results for the 93 MVA power transformer of reference [21] are shown in Figure 10. In this case we have only a small number of plotted test points due, in addition to the low power factor, also to the poor resolution of the measurements especially at low and high frequencies. However, the number of points extracted from the tests is adequate to assess the validity of the model. The simulation shown was obtained with a model with 20 sections. The results are satisfactory in a global, rather than detailed sense, as they permit the representation of strongly increased damping in transient phenomena involving fast oscillations. We will show in the next section that in time domain simulations we obtain very good correlation with measurements since there is an averaging process when the transient contains several frequencies.

![Graph showing frequency variation of input resistance for the 93 MVA transformer.](image)

Figure 9. Frequency variation of input resistance for the 75 kVA transformer

a) Capacitances to tank neglected
b) Capacitances to tank considered

The above frequency domain simulations took less than one minute of cpu time on a MIPS - RISC 6200 computer. To speed up the calculation of parameters (including the R-matrix) and still use the procedures of reference [2] we have grouped several turns into sections and have performed the calculations for sections. We have followed the procedure described in Appendix C for the calculation of parameters for the 75 kVA and the 93 MVA transformer.

### Time Domain

The 75 kVA transformer was subjected to a periodic double exponential impulse voltage \( v = e^{-t/T} - e^{-t/\alpha} \) as shown in Figure 11a (with a period \( T = 0.1 \) and \( \alpha = 0.01 \)). Thus, the results can be synthesized by Fourier methods or obtained by numerical integration in the time domain. The former was necessary for processing the data available from tests. In Figure 11b we show a full period of the currents obtained using the test data and simulations when the effect of eddy currents is neglected. We note that the transient current does not have adequate damping. The result of including in the model the increase of the resistance due to the eddy currents is presented in Figure 11c. We can see that the time domain results when the effects of eddy currents are included are remarkably good although the frequency response has shown not insignificant differences.

In Figure 12 we show the transient current when we apply a double exponential voltage to the 93 MVA transformer. Although the frequency response of Figure 10 showed some differences the transient response of the model is in good agreement with the measurements.

In reference [4] we have presented the overvoltages produced when chopping the current in the secondary of the small transformer during a short circuit test. The simulation was performed assuming equal currents in all turns (internal capacitances were not considered). With the model presented in this paper we are able to take into account the inter-turn (inter-section) capacitances as well as the damping produced by the eddy currents in the windings of the transformer. In Figure 13 we show the simulation results for the energization of the transformer with its secondary short circuited. In the figures we are plotting the voltage at the center of the primary winding. Figure 13a shows the results when the eddy current losses are included in the simulation and Figure 13b when they are neglected (\( R = \text{constant} \)). The difference in attenuation is quite significant.

Note that the model has the very attractive feature of permitting to display voltages and currents at, or between, virtually any points in the windings.
CONCLUSIONS

A state space model for the eddy current losses in the windings of transformers has been developed. This model is suitable for the study of high frequency electromagnetic transients since it is based on turn-to-turn information. The location of the losses (for hot spot calculations) can be obtained from the matrices constituting the total resistance matrix. A fitting technique for the approximation of smooth transfer function matrices has also been developed. Its applicability reaches beyond the field of transformer modeling.

For validation, the total eddy current losses for the turn-to-turn model have been compared with the losses obtained by a well-tested formula for full windings. Frequency and time domain simulations have been compared against test results for illustration and further validation.

By performing calculations with sections containing groups of turns, the computations become sufficiently fast for practical applications even for large transformers.

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REFERENCES

APPENDIXES

Appendix A. Expressions for the Magnetic Field for the Image Method

The expressions for \( f^2 \) and \( f^4 \) in equations (6a) and (6b) are presented in this appendix. We have used for the image current \( I_m = 2.5 I \) according to the conclusion obtained in reference [2].

\[
f^2 = \frac{1}{2r} \frac{2}{\sqrt{r^2 + r^2 + z^2}} \left[ K(k) - r^2 + r^2 + z^2 \right] E(k) + \frac{2.5}{2r} \frac{2}{\sqrt{r^2 + r^2 + z^2}} \left[ K(k_m) - r^2 + r^2 + z^2 \right] E(k_m)
\]

and

\[
f^4 = \frac{1}{2r} \frac{2}{\sqrt{(r^2 + r^2 + z^2)^2}} \left[ K(k) - \frac{r^2 - r^2 - z^2}{(r^2 + r^2 + z^2)^2} \right] E(k) + \frac{2.5}{2r} \frac{2}{\sqrt{(r^2 + r^2 + z^2)^2}} \left[ K(k_m) - \frac{r^2 - r^2 - z^2}{(r^2 + r^2 + z^2)^2} \right] E(k_m)
\]

where

\[ k_m = \sqrt{\frac{4r_m r}{r^2 + r^2 + z^2}} \]

\[ r_m = \text{radius of the image conductor} \]

Appendix B. Detailed Definition of the State Equation Matrices

The variables used in equations (32) and (33) are the following:

\[ x = [x_{11}, x_{21}, \ldots, x_{1n}, x_{21}, \ldots, x_{n}, y_{1}, y_{2}, \ldots, y_{k}] \]

\[ I_p = [i_1, i_2, \ldots, i_p] \]

\[ V_p = [v_1, v_2, \ldots, v_p] \]

\[ A = \text{diag} \left[ \frac{L_{11}}{L_{11}}, \frac{L_{12}}{L_{12}}, \ldots, \frac{L_{1n}}{L_{1n}}, \frac{L_{21}}{L_{21}}, \ldots, \frac{L_{2n}}{L_{2n}}, \ldots, \frac{L_{nn}}{L_{nn}} \right] \]