EL736 Communications Networks II: Design and Algorithms

Class9: Fair Networks

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Outline

- Fair sharing of network resource
- Max-min Fairness
- Proportional Fairness
- Extension

Fair Networks

- Elastic Users:
  - demand volume NOT fixed
  - greedy users: use up resource if any, e.g. TCP
  - competition resolution?
- Fairness: how to allocate available resource among network users.
  - capacitated design: resource=bandwidth
  - uncapacitated design: resource=budget
- Applications
  - rate control
  - bandwidth reservation
  - link dimensioning

Max-Min Fairness: definition

- Lexicographical Comparison
  - a n-vector $x=(x_1,x_2, \ldots,x_n)$ sorted in non-decreasing order $(x_1 \le x_2 \le \ldots \le x_n)$ is lexicographically greater than another n-vector $y=(y_1,y_2, \ldots,y_n)$ sorted in non-decreasing order if an index k, $0 \le k \le n$ exists, such that $x_i = y_i$, for $i=1,2,\ldots,k-1$ and $x_k > y_k$
  - $(2,4,5) >_L (2,3,100)$
- Max-min Fairness: an allocation is max-min fair if its lexicographically greater than any feasible allocation

- Uniqueness?

Other Fairness Measures

Proportional fairness [Kelly, Maulloo & Tan, '98]

- A feasible rate vector x is proportionally fair if for every other feasible rate vector y

$$\sum w_i \frac{(y_i - x_i)}{x_i} \leq 0$$

- Proposed decentralized algorithm, proved properties

Generalized notions of fairness [Mo & Walrand, 2000]

- (α, p)-proportional fairness: A feasible rate vector x is fair if for any other feasible rate vector y

$$\sum w_i \frac{(y_i - x_i)}{x_i^{\alpha}} \leq 0$$

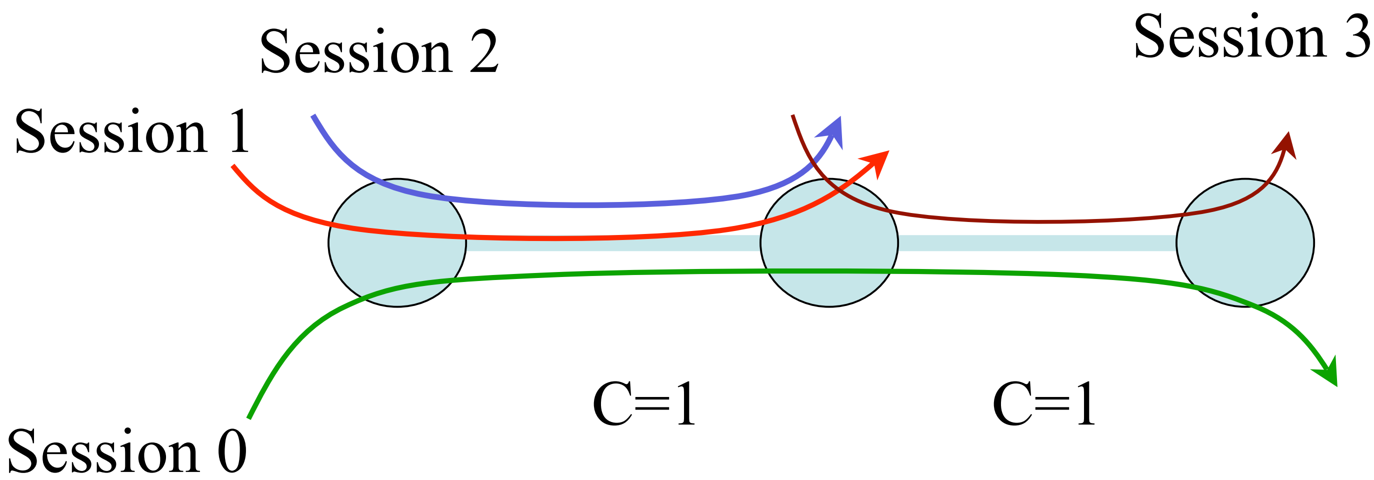
- Special cases: $\alpha = 1$: proportional fairness
  $\alpha \to \infty$: max-min fairness

Capacitated Max-Min Flow Allocation

- Fixed single path for each demand

- Proposition: a flow allocation is max-min fair if for each demand d there exists at least one bottle-neck, and at least on one of its bottle-necks, demand d has the highest rate among all demands sharing that bottle-neck link.

Max-min Fairness Example



- Max-min fair flow allocation
  - sessions 0,1,2: flow rate of 1/3
  - session 3: flow rate of 2/3

Max-Min Fairness: other definitions

- Definition1: A feasible rate vector R^* is max-min fair if no rate R_i can be increased without decreasing some R_k s.t. $R_k < R_i$

- Definition2: A feasible rate vector R^* is an optimal solution to the MaxMin problem iff for every feasible rate vector $\hat{R}$ with $\hat{R}_i > R_i^*$, for some user i, then there exists a user k such that $\hat{R}_k < R_k^*$ and $\hat{R}_k < \hat{R}_i$

How to Find Max-min Fair Allocation?

- Idea: equal share as long as possible
- Procedure
  1. start with 0 rate for all demand
  2. increase rate at the same speed for all demands, until some link saturated
  3. remove saturated links, and demands using those links
  4. go back to step 2 until no demand left.

Max-min Fair Algorithm

ALGORITHM 8.1 Algorithm for Solving A/MMF/EFIXSP

Step 0: Put $\boldsymbol{x}^* = \mathbf{0}$.

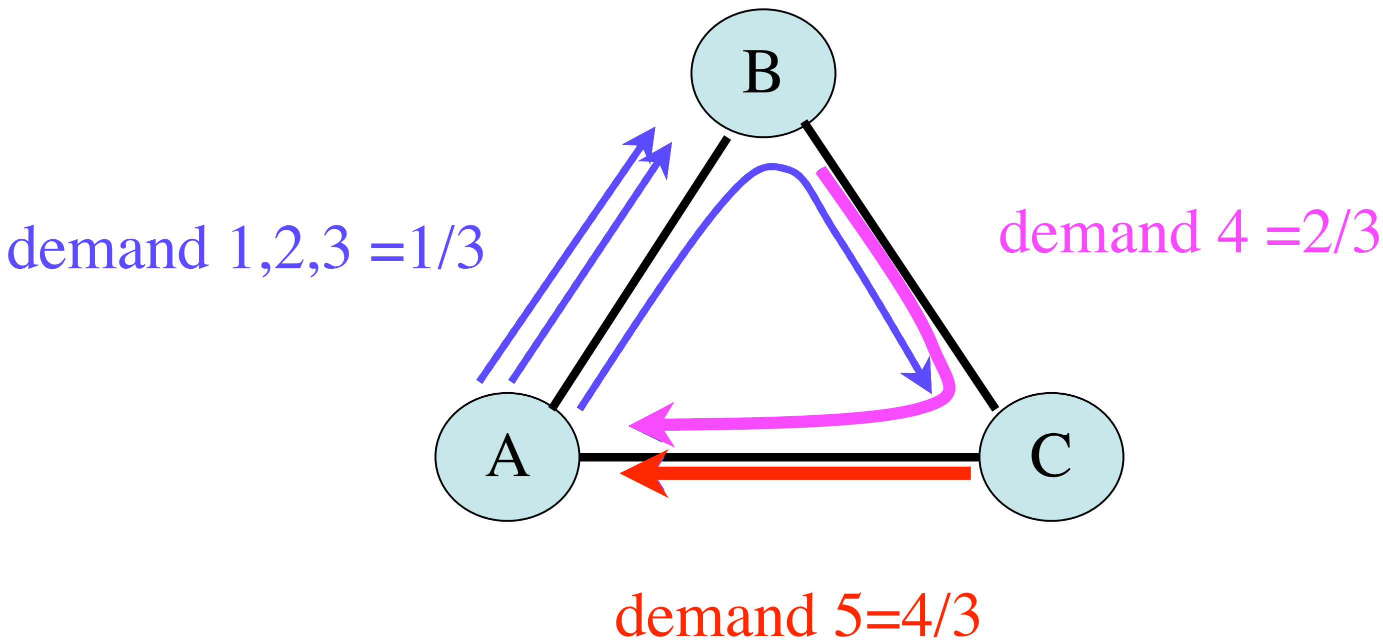
Step 1: $t := \min\{c_e / \sum_d \delta_{ed} : e = 1, 2, \ldots, E\}$.

Step 2: $c_e := c_e - t(\sum_d \delta_{ed})$ for $e = 1, 2, \ldots, E$; $x_d^* := x_d^* + t$ for $d = 1, 2, \ldots, D$. Remove all saturated links (all e with $c_e = 0$). For each removed link e remove all paths and corresponding demands that use the removed link (all d with $\delta_{ed} = 1$).

Step 3: If there are no demands left then stop, otherwise go to Step 1.

Max-min Fair Example

link rate: AB=BC=1, CA=2



Extended MMF

- lower and upper bound on demands
- weighted demand rate

A/MMF/EFSP **Link-Path Formulation**

Extended MMF/EFIXSP

indices

$d = 1, 2, ..., D$ demands

$e = 1, 2, ..., E$ links

constants

δ_{ed} = 1 if link e belongs to the fixed path of demand d; 0, otherwise

c_e capacity of link e

w_d weight of demand d

h_d lower bound for the flow of demand d

H_d upper bound for the flow of demand d

variables

x_d flow assigned to demand d, $x = (x_1, x_2, \ldots, x_D)$

objective

find allocation vector x which, when sorted in non-decreasing order, is lexicographically maximal among all allocation vectors sorted in non-decreasing order

constraints

$$h_d \le x_d \le H_d, \quad d = 1, 2, \ldots, D \quad (8.1.4a)$$

$$\sum_d \delta_{ed} w_d x_d \le c_e, \quad e = 1, 2, \ldots, E \quad (8.1.4b)$$

$$x \ge 0. \quad (8.1.4c)$$

Extended MMF: algorithm

ALGORITHM 8.2 Algorithm for Solving A/MMF/EFIXSP

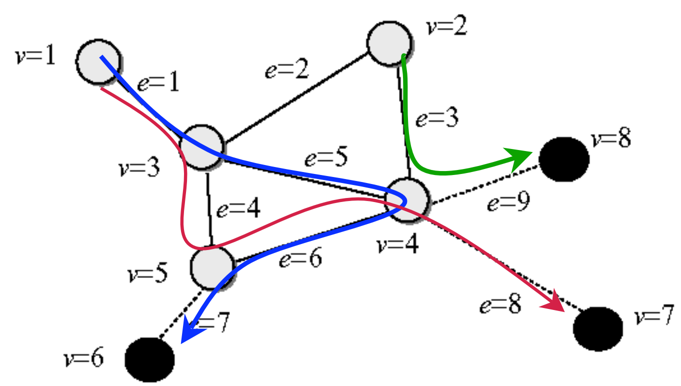
Step 0:	Put $x_d^* = h_d$ for $d = 1, 2, \ldots, D$ and $c_e := c_e - \sum_d \delta_{ed} w_d h_d$ for $e = 1, 2, \ldots, E$. Remove all saturated links (all e with $c_e = 0$). For each removed link e remove all paths and corresponding demands that use the removed link (all d with $\delta_{ed} = 1$).
Step 1:	Solve the LP Problem (8.1.5) to obtain t.
Step 2:	$c_e := c_e - t(\sum_d w_d \delta_{ed})$ for $e = 1, 2, \ldots, E$; $x_d^* := x_d^* + t$ for $d = 1, 2, \ldots, D$. Remove all saturated links (all e with $c_e = 0$). For each removed link e remove all paths and corresponding demands that use the removed link (all d with $\delta_{ed} = 1$).
Step 3:	If there are no demands left then stop, otherwise go to Step 1.

maximize t (8.1.5a)

subject to $t(\sum_d w_d \delta_{ed}) \leq c_e, \quad e = 1, 2, \ldots, E.$ (8.1.5b)

Deal with Upper Bound

- Add one auxiliary virtual link with link capacity $w_d H_d$ for each demand with upper bound H_d



$c_7 = w_1 H_1$, $c_8 = w_2 H_2$, $c_9 = w_3 H_3$
d=1 between v=1 and v=5, path {1,5,6}: leaf node v=6, new path {1,5,6,7}
d=2 between v=1 and v=4, path {1,4,6}: leaf node v=7, new path {1,4,6,8}
d=3 between v=2 and v=4, path {3}: leaf node v=8, new path {3,9}

FIGURE 8.2 Augmented Network

MMF with Flexible Paths

- one demand can take multiple paths
- max-min over aggregate rate for each demand
- potentially more fair than single-path only
- more difficult to solve

constants

δ_{edp} = 1 if link e belongs to path p of demand d; 0, otherwise

c_e capacity of link e

variables

x_{dp} flow (bandwidth) allocated to path p of demand d

X_d total flow (bandwidth) allocated to demand d, $\boldsymbol{X} = (X_1, X_2, \ldots, X_D)$

objective

find total flow allocation vector $\boldsymbol{X}$ which, when sorted in non-decreasing order, is lexicographically maximal among all total allocation vectors sorted in non-decreasing order.

constraints

$$\textstyle\sum_p x_{dp} = X_d \qquad d = 1, 2, \ldots, D \qquad (8.2.1a)$$

$$\textstyle\sum_d \sum_p \delta_{edp} x_{dp} \le c_e \qquad e = 1, 2, \ldots, E \qquad (8.2.1b)$$

$$\text{all } x_{dp} \ge 0. \qquad (8.2.1c)$$

Uncapacitated Problem

- Max-min fair sharing of budget
- Formulation

***LP:* D/MMF/FLMP** **Link-Path Formulation**

MMF With Flexible Multiple Paths

indices

- $d = 1, 2, ..., D$ demands
- $p = 1, 2, \ldots, P_d$ candidate paths for demand d
- $e = 1, 2, ..., E$ links

constants

- δ_{edp} = 1 if link e belongs to path p of demand d; 0, otherwise
- ξ_e unit cost of link e
- ζ_d = $\sum_e \xi_e \delta_{edp(d)}$ - cost (length) of the shortest candidate path, denoted by $p(d)$, realizing demand d
- B assumed budget

variables

- x_{dp} flow (bandwidth) allocated to path p of demand d
- X total flow allocated to demand d (the same for all demands)
- y_e capacity of link e

objective

$$\text{maximize } X \qquad (8.2.14a)$$

constraints

$$\sum_e \xi_e y_e \le B \qquad (8.2.14b)$$

$$\sum_p x_{dp} = X, \qquad d = 1, 2, \ldots, D \qquad (8.2.14c)$$

$$\sum_d \sum_p \delta_{edp} x_{dp} \le y_e, \qquad e = 1, 2, \ldots, E \qquad (8.2.14d)$$

$$\text{all } x_{dp} \ge 0. \qquad (8.2.14e)$$

Uncapacitated Problem

- max-min allocation
  - all demands have the same rate
  - each demand takes the shortest path
- proof?

Proportional Fairness

- Proportional Fairness [Kelly, Maulloo & Tan, '98]
  - A feasible rate vector x is proportionally fair if for every other feasible rate vector y

$$\sum w_i \frac{(y_i - x_i)}{x_i} \leq 0$$

- formulation

CXP: **A/PF/FIXSP** | **Link-Path Formulation**
PF with Fixed Single Paths
variables

x_d flow assigned to demand d, $\boldsymbol{x} = (x_1, x_2, \ldots, x_D)$

objective

maximize $F(\boldsymbol{x}) = \sum_d \log x_d$ (8.1.6a)

constraints

$\sum_d \delta_{ed} x_d \leq c_e, \quad e = 1, 2, \ldots, E$ (8.1.6b)

$\boldsymbol{x} \geq 0.$ (8.1.6c)

Linear Approximation of PF

***LP:* A/PF/LAFIXSP** **Link-Path Formulation**

Linear Approximation of A/PF/FIXSP

indices

$d = 1, 2, \ldots, D$ demands

$e = 1, 2, \ldots, E$ links

$k = 1, 2, \ldots, K$ consecutive pieces of the approximation of $\log x$

constants

δ_{ed} = 1 if link e belongs to the fixed path of demand d; 0, otherwise

c_e capacity of link e

a_k, b_k coefficients of the linear pieces of the linear approximation of $\log x$

variables

x_d flow assigned to demand d, $\boldsymbol{x} = (x_1, x_2, \ldots, x_D)$

f_d approximation of $\log x_d$

objective

$$\text{maximize } \boldsymbol{F} = \textstyle\sum_d f_d \quad (8.1.8a)$$

constraints

$$f_d \le a_k x_d + b_k, \quad d = 1, 2, \ldots, D \quad k = 1, 2, \ldots, K \quad (8.1.8b)$$

$$\textstyle\sum_d \delta_{ed} x_d \le c_e, \quad e = 1, 2, \ldots, E \quad (8.1.8c)$$

$$\boldsymbol{x} \ge 0. \quad (8.1.8d)$$

Extended PF Formulation

CXP: **A/PF/EFIXSP** **Link-Path Formulation**

Extended A/PF/FIXSP

variables

x_d flow assigned to demand d, $\boldsymbol{x} = (x_1, x_2, \ldots, x_D)$

objective

$$\text{maximize } \boldsymbol{F}(\boldsymbol{x}) = \textstyle\sum_d w_d \log x_d \qquad (8.1.9a)$$

constraints

$$h_d \le x_d \le H_d, \qquad d = 1, 2, \ldots, D \qquad (8.1.9b)$$

$$\textstyle\sum_d \delta_{ed} x_d \le c_e, \qquad e = 1, 2, \ldots, E \qquad (8.1.9c)$$

$$\boldsymbol{x} \ge 0. \qquad (8.1.9d)$$

Uncapacitated PF Design

- maximize network revenue minus investment

XX: D/PF/EOUF **Link-Path Formulation**

Extended Objective and Unbounded Flows

additional constant

B_0 upper bound for the cost of links (in general different than budget B in D/PF/BCUF)

variables

x_d flow (bandwidth) allocated to the path of demand d, $\boldsymbol{x} = (x_1, x_2, \ldots, x_D)$

y_e capacity of link e, $\boldsymbol{y} = (y_1, y_2, \ldots, y_E)$

objective

$$\text{maximize } \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \sum_d w_e \log x_d - \sum_e \xi_e y_e \qquad (8.3.30a)$$

constraints

$$\sum_e \xi_e y_e \leq B_0 \qquad (8.3.30b)$$

$$\sum_d \delta_{ed} x_d \leq y_e, \quad e = 1, 2, \ldots, E \qquad (8.3.30c)$$

$$\boldsymbol{x}, \boldsymbol{y} \geq 0. \qquad (8.3.30d)$$