EL736 Communications Networks II: Design and Algorithms

Class8: Networks with Shortest-Path Routing

Yong Liu

10/31/2007

Polytechnic University

Celebrating 150 Years in Brooklyn

Outline

- Shortest-path routing
- MIP Formulation
- Duality and Shortest-Path Routing
- Heuristic Method for link weights
- Examples
- Extensions

Shortest-path Routing

- Take the shortest-path(s) from one point to the other
  - path length = summation of link weights
  - algorithm: Dijkstra, Bellman-Ford, extensions,
  - intra-domain routing: link state: OSPF, IS-IS
  - equal-cost multi-path split (ECMP)
- Intra-domain Traffic Engineering
  - Good end-to-end performance for users
  - Efficient use of the network resources
  - Reliable system even in the presence of failures

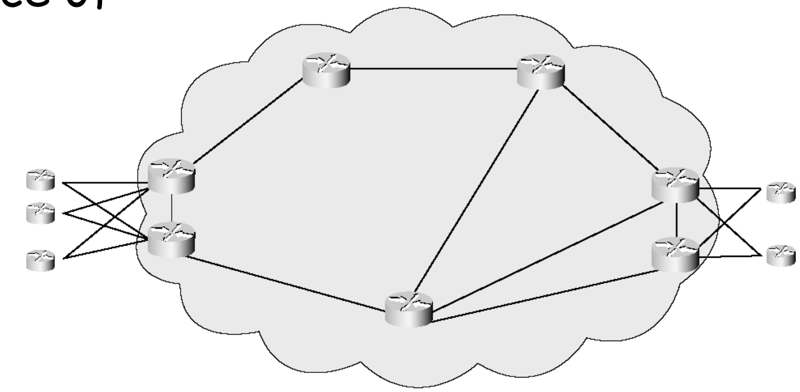
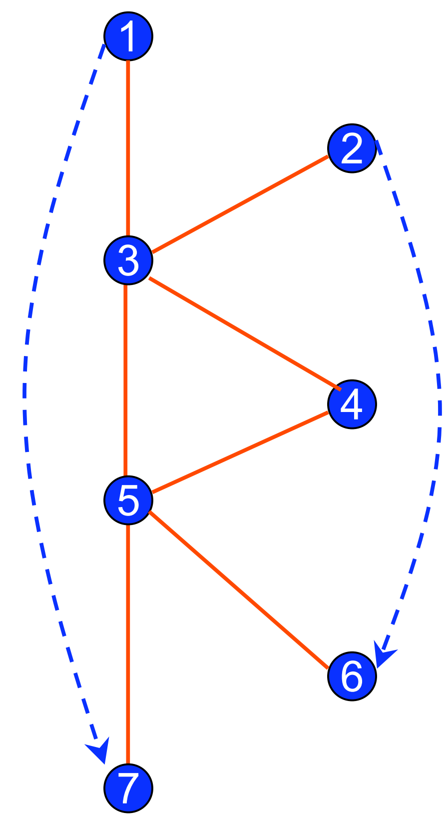


FIGURE 3.1 Intra-Domain IP Network

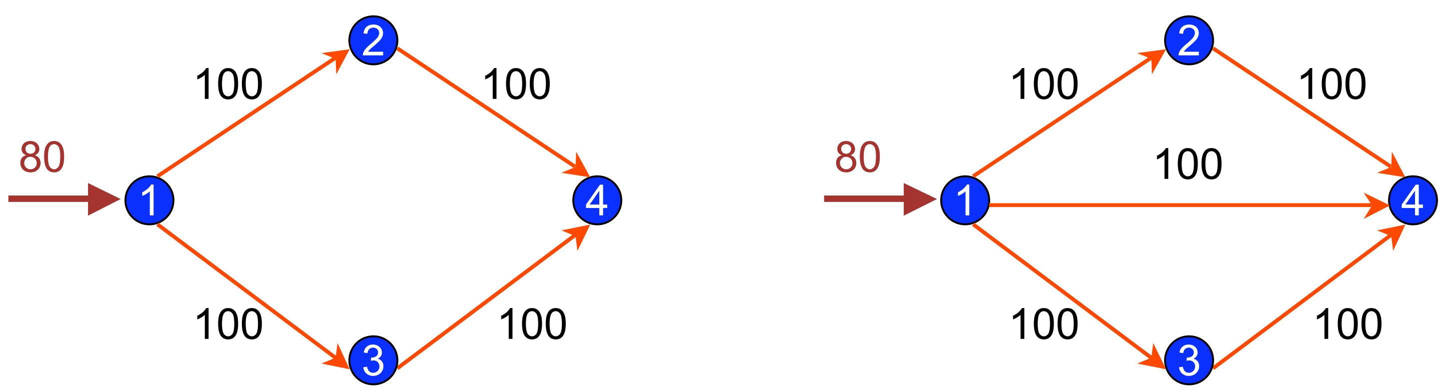
TE Optimization: The Problem

- Intra-domain Traffic Engineering
  - Predict influence of weight changes on traffic flow
  - Minimize objective function (say, of link utilization)
- Inputs
  - Networks topology: capacitated, directed graph
  - Routing configuration: routing weight for each link
  - Traffic matrix: offered load each pair of nodes
- Outputs
  - Shortest path(s) for each node pair
  - Volume of traffic on each link in the graph
  - Value of the objective function



Which link weight system to use

- Link Weight can be
  - 1, (hop count)
  - propagation delay (const.)
  - 1/C (Cisco)
  - congestion delay (load sensitive, online update)
- Objective dependent choice
  - hop count vs. congestion delay
  - ECMP vs. equal delay routing



Shortest Path Routing: bounded link delay

ECMP Shortest-Path Routing: Bounded Link Delay

- **indices**
  - $d = 1, 2, ..., D$ demands
  - $p = 1, 2, ..., P_d$ paths for demand d (all simple paths in the network graph)
  - $e = 1, 2, ..., E$ links
- **constants**
  - h_d volume of demand d
  - δ_{edp} $= 1$ if link e belongs to path p realizing demand d; 0, otherwise
  - c_e capacity of link e
  - γ_e link utilization factor for link e ($\gamma_e = 1 - 1/(c_e T)$)
- **variables**
  - w_e metric of link e, $\boldsymbol{w} = (w_1, w_2, ..., w_E)$
  - $x_{dp}(\boldsymbol{w})$ (non-negative) flow induced by link metric system $\boldsymbol{w}$ for demand d on path p
- **constraints**

$$\sum_p x_{dp}(\boldsymbol{w}) = h_d, \quad d = 1, 2, ..., D$$

$$\sum_d \sum_p \delta_{edp} x_{dp}(\boldsymbol{w}) \le \gamma_e c_e, \quad e = 1, 2, ..., E$$

$$\boldsymbol{w} \in \boldsymbol{W}.$$

Penalty Function

- use link penalty function to replace link constraints

ECMP shortest-path routing: Penalty Function

- **indices**
  - $d = 1, 2, ..., D$ demands
  - $p = 1, 2, ..., P_d$ paths for demand d (all simple paths in the network graph)
  - $e = 1, 2, ..., E$ links
- **constants**
  - h_d volume of demand d
  - δ_{edp} $= 1$ if link e belongs to path p realizing demand d; 0, otherwise
- **variables**
  - w_e metric of link e, $\boldsymbol{w} = (w_1, w_2, ..., w_E)$
  - $x_{dp}(\boldsymbol{w})$ flow induced by link metric system $\boldsymbol{w}$ for demand d on path p
  - $\underline{y}_e$ load of link e
- **objective**

$$\boldsymbol{F} = \sum_e f_e(\underline{y}_e)$$

- **constraints**

$$\sum_p x_{dp}(\boldsymbol{w}) = h_d, \quad d = 1, 2, ..., D$$

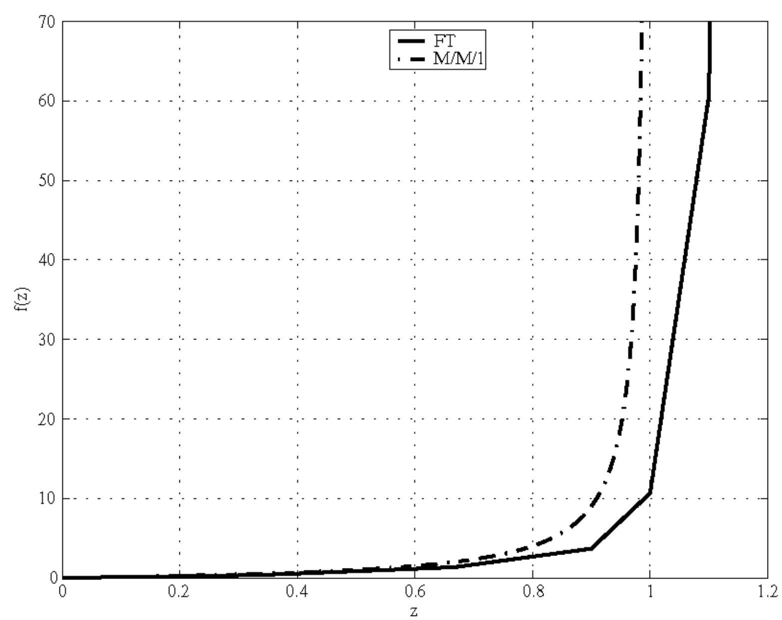
$$\sum_d \sum_p \delta_{edp}\, x_{dp}(\boldsymbol{w}) = \underline{y}_e, \quad e = 1, 2, ..., E$$

$$\boldsymbol{w} \in \boldsymbol{W}.$$

Shortest Path Routing: minimum average delay

- load sensitive link delay
  - $$f_e(\underline{y}_e) = \frac{\underline{y}_e}{c_e - \underline{y}_e}$$
- piece-wise linear approximation

$$f_e(\underline{y}_e) = \begin{cases} \underline{y}_e & \text{for} \quad 0 \le \underline{y}_e/c_e < 1/3 \\ 3\,\underline{y}_e - \frac{2}{3}\,c_e & \text{for} \quad 1/3 \le \underline{y}_e/c_e < 2/3 \\ 10\,\underline{y}_e - \frac{16}{3}\,c_e & \text{for} \quad 2/3 \le \underline{y}_e/c_e < 910 \\ 70\,\underline{y}_e - \frac{178}{3}\,c_e & \text{for} \quad 9/10 \le \underline{y}_e/c_e < 1 \\ 500\,\underline{y}_e - \frac{1{,}468}{3}\,c_e & \text{for} \quad 1 \le \underline{y}_e/c_e < 11/10 \\ 5{,}000\,\underline{y}_e - \frac{16{,}318}{3}\,c_e & \text{for} \quad 11/10 \le \underline{y}_e/c_e < \infty. \end{cases}$$



Shortest Path Routing: minimum average delay

- **objective**

  minimize $\boldsymbol{F} = \sum_e r_e$

- **constraints**

$$
\begin{aligned}
&\textstyle\sum_p x_{dp}(\boldsymbol{w}) = h_d && d = 1, 2, \ldots, D \\
&\textstyle\sum_d \sum_p \delta_{edp} x_{dp}(\boldsymbol{w}) = \underline{y}_e, && e = 1, 2, \ldots, E \\
&r_e \geq \underline{y}_e, && e = 1, 2, \ldots, E \\
&r_e \geq 3\,\underline{y}_e - \tfrac{2}{3}\,c_e, && e = 1, 2, \ldots, E \\
&r_e \geq 10\,\underline{y}_e - \frac{16}{3}\,c_e, && e = 1, 2, \ldots, E \\
&r_e \geq 70\,\underline{y}_e - \frac{178}{3}\,c_e, && e = 1, 2, \ldots, E \\
&r_e \geq 500\,\underline{y}_e - \frac{1{,}468}{3}\,c_e, && e = 1, 2, \ldots, E \\
&r_e \geq 5{,}000\,\underline{y}_e - \frac{16{,}318}{3}\,c_e, && e = 1, 2, \ldots, E \\
&\boldsymbol{w} \in \boldsymbol{W}.
\end{aligned}
$$

Minimization of Maximum Link Utilization

Minimization of Maximum Link Utilization

- **indices**

  $d = 1, 2, ..., D$ demands
  $p = 1, 2, ..., P_d$ paths for demand d (all simple paths in the network graph)
  $e = 1, 2, ..., E$ links

- **constants**

  h_d volume of demand d
  δ_{edp} $= 1$ if link e belongs to path p realizing demand d; 0, otherwise
  γ_e link-based adjustment for acceptable link utilization

- **variables**

  w_e metric of link e, $\boldsymbol{w} = (w_1, w_2, ..., w_E)$
  $x_{dp}(\boldsymbol{w})$ flow induced by link metric system $\boldsymbol{w}$ for demand d on path p
  r maximum link utilization variable

- **objective**

  minimize $\boldsymbol{F} = r$

- **constraints**

  $\sum_p x_{dp}(\boldsymbol{w}) = h_d, \quad d = 1, 2, ..., D$
  $\sum_d \sum_p \delta_{edp} x_{dp}(\boldsymbol{w}) \leq \gamma_e c_e\, r, \quad e = 1, 2, ..., E$
  $\boldsymbol{w} \in \boldsymbol{W}.$

MIP Formulation

ECMP Shortest-Path Routing

- **indices**

  $t, v, s = 1, 2, ..., V$ nodes

  $e = 1, 2, ..., E$ links

- **constants**

  h_{vt} demand volume from node v to node t

  $i(e)$ starting (originating) node of link e

  $j(e)$ terminating node of link e

  M large number

  c_e capacity of link e

- **variables**

  w_e metric of link e

  r_{vt} length of the shortest-path from v to t $(v \neq t)$

  x_{et} flow to node t on link e

  y_{vt} common value of non-zero flow from node v to node t assigned to links outgoing from v and belonging to the shortest-paths from v to t

  u_{et} binary variable equal to 1 if and only if link e is on a shortest-path to node t

MIP Formulation

ECMP Shortest-Path Routing (Contd.)

- **constraints**

$\sum_{\{e:j(e)=t\}} x_{et} = \sum_{s \neq t} h_{st},$	$t = 1, 2, ..., V$	
$\sum_{\{e:i(e)=v\}} x_{et} - \sum_{\{e:j(e)=v\}} x_{et} = h_{vt},$	$t = 1, 2, ..., V$	$v = 1, 2, ..., V, v \neq t$
$\sum_{t} x_{et} \leq c_e,$	$e = 1, 2, ..., E$	
$0 \leq y_{i(e)t} - x_{et} \leq (1 - u_{et}) \sum_{v} h_{vt},$	$t = 1, 2, ..., V$	$e = 1, 2, ..., E$
$x_{et} \leq u_{et} \sum_{v} h_{vt},$	$t = 1, 2, ..., V$	$e = 1, 2, ..., E$
$0 \leq r_{j(e)t} + w_e - r_{i(e)t} \leq (1 - u_{et}) M,$	$t = 1, 2, ..., V$	$e = 1, 2, ..., E$
$1 - u_{et} \leq r_{j(e)t} + w_e - r_{i(e)t},$	$t = 1, 2, ..., V$	$e = 1, 2, ..., E$
$w_e \geq 1,$	$e = 1, 2, ..., E.$	

Duality: Lagrangian

standard form problem (not necessarily convex)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

variable $x \in \mathbf{R}^n$, domain $\mathcal{D}$, optimal value $p^\star$

Lagrangian: $L : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$, with $\mathbf{dom}\, L = \mathcal{D} \times \mathbf{R}^m \times \mathbf{R}^p$,

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

- weighted sum of objective and constraint functions
- λ_i is Lagrange multiplier associated with $f_i(x) \leq 0$
- ν_i is Lagrange multiplier associated with $h_i(x) = 0$

Slides from Convex Optimization, Boyd & Vandenberghe

Duality: dual function

Lagrange dual function: $g : \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$,

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \\ &= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right) \end{aligned}$$

g is concave, can be $-\infty$ for some λ, ν

lower bound property: if $\lambda \succeq 0$, then $g(\lambda, \nu) \leq p^\star$

proof: if $\tilde{x}$ is feasible and $\lambda \succeq 0$, then

$$f_0(\tilde{x}) \geq L(\tilde{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = g(\lambda, \nu)$$

minimizing over all feasible $\tilde{x}$ gives $p^\star \geq g(\lambda, \nu)$

Slides from Convex Optimization, Boyd & Vandenberghe

Dual Problem

Lagrange dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- finds best lower bound on $p^\star$, obtained from Lagrange dual function
- a convex optimization problem; optimal value denoted $d^\star$
- λ, ν are dual feasible if $\lambda \succeq 0$, $(\lambda, \nu) \in \mathbf{dom}\, g$
- often simplified by making implicit constraint $(\lambda, \nu) \in \mathbf{dom}\, g$ explicit

Slides from Convex Optimization, Boyd & Vandenberghe

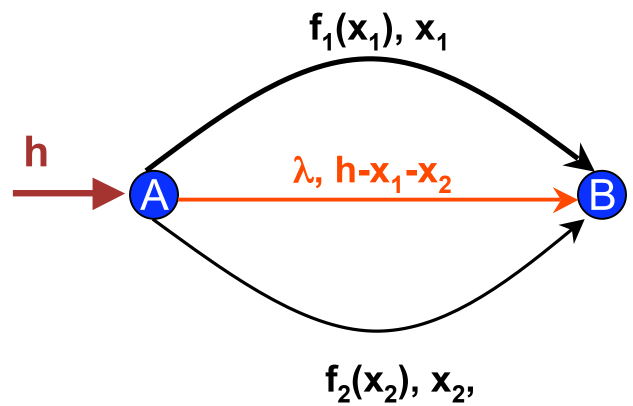
Duality Theorem

- Weak Duality: $d^* \leq p^*$
  - always hold (convex, non-convex problems)
  - find non-trivial lower bounds for complex problems
  - duality gap: $p^* - d^*$
- Strong Duality: $d^* = p^*$
  - does not hold in general
  - hold for most convex problems, (including LP)
  - zero duality gap, obtain optimal solution for the original problem by solving the dual problem.
- Advantages of working with Duals
  - less constraints
  - decoupling
  - distributed algorithms:
    - distributed routing algorithms
    - end system congestion control, TCP

Duality: routing example

- minimal delay routing

$$F^* = \min_{x_1, x_2 \geq 0} \quad x_1 f_1(x_1) + x_2 f_2(x_2)$$
$$\text{subject to } x_1 + x_2 = h$$



- Lagrange dual function

$$g(\lambda) = \min_{x_1, x_2 \geq 0} \quad x_1 f_1(x_1) + x_2 f_2(x_2) + \lambda(h - x_1 - x_2)$$

$$g(\lambda) \leq F^*?$$

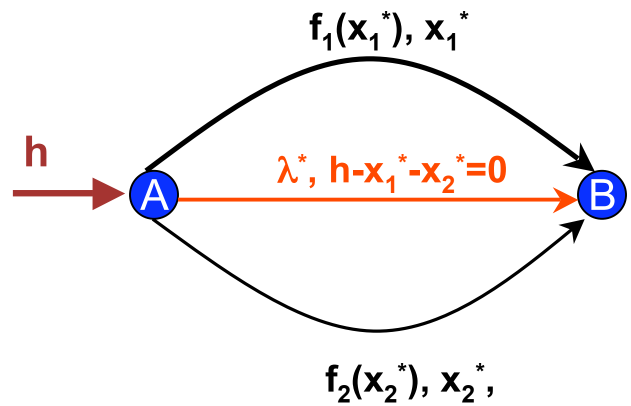
- Decoupling

$$g_1(\lambda) = \min_{x_1 \geq 0} x_1 f_1(x_1) - \lambda x_1$$
$$g_2(\lambda) = \min_{x_2 \geq 0} x_2 f_2(x_2) - \lambda x_2$$
$$g(\lambda) = g_1(\lambda) + g_2(\lambda) + \lambda h$$

Duality: routing example

- Dual Problem

$$G^* = \max_{\lambda} g(\lambda)$$



- Strong Duality

$$F^* = G^*$$

$$x_1^* = \underset{x_1 \geq 0}{\text{argmin}}\ x_1 f_1(x_1) - \lambda^* x_1$$

$$x_2^* = \underset{x_2 \geq 0}{\text{argmin}}\ x_2 f_2(x_2) - \lambda^* x_2$$

$$\frac{d}{dx_1} x_1 f_1(x_1)|_{x_1 = x_1^*} = \frac{d}{dx_2} x_2 f_2(x_2)|_{x_2 = x_2^*} = \lambda^*$$

- Dual algorithm:
  - increase delay on virtual link if $x_1+x_2<h$, decrease delay otherwise

Routing Duality: generalization

- multi-demand/multi-path

$$\min_{x_{dp} \geq 0} \sum_e f_e\Big(\sum_d \sum_p x_{dp}\Big)$$

subject to $\sum_p x_{dp} = h_d,\ d = 1, 2, \cdots, D$

- routing duality

$$y_e^* = \sum_d \sum_p x_{dp}^*$$

link weight $w_e \triangleq f_e'(y_e^*)$

path length $l_{dp} \triangleq \sum_e \delta_{edp} w_e \geq \lambda_d^*$

$x_{dp} > 0$ iff $l_{dp} = \lambda_d^*$

- optimal flows only on shortest-paths!

Duality and Shortest-path Routing

Residual Capacity Maximization

- **indices**

  $d = 1, 2, ..., D$ demands

  $p = 1, 2, ..., P_d$ paths for demand d (all simple paths in the network graph)

  $e = 1, 2, ..., E$ links

- **constants**

  h_d volume of demand d

  δ_{edp} $= 1$ if link e belongs to path p realizing demand d; 0, otherwise

  c_e capacity of link e

  γ_e link adjustment factor of link e

  b_e value of one unit of idle capacity on link e

- **variables**

  x_{dp} non-negative flow on path p for demand d

- **objective**

  maximize $\boldsymbol{F} = \sum_e b_e(\gamma_e c_e - \sum_d \sum_p \delta_{edp} x_{dp})$

- **constraints**

  $\sum_p x_{dp} = h_d, \quad d = 1, 2, ...D$ (7.2.2b)

  (1)

  $\sum_d \sum_p \delta_{edp} x_{dp} \leq \gamma_e c_e, \quad e = 1, 2, ..., E.$ (7.2.2c)

Dual Formulation

LP Dual of the above problem

- **variables**

  λ_d (unrestricted) dual multiplier associated with constraints(7.2.2b)

  π_e non-negative dual multiplier associated with constraints(7.2.2c)

- **objective**

  maximize $G(\boldsymbol{\pi}, \boldsymbol{\lambda}) = \sum_d \lambda_d h_d - \sum_e (b_e + \pi_e)\gamma_e c_e$

- **constraints**

  $\lambda_d \leq \sum_e \delta_{edp}(b_e + \pi_e), \quad p = 1, 2, ..., P_d, \quad d = 1, 2, ..., D.$

Duality

$$\lambda_d^* = \min_p \{\sum_e \delta_{edp}(b_e + \pi_e^*)\}, \quad d = 1, 2, \cdots, D$$

link weight $w_e^* \triangleq b_e + \pi_e^*$

- optimal flows only on shortest-paths!

Optimal Link Weights

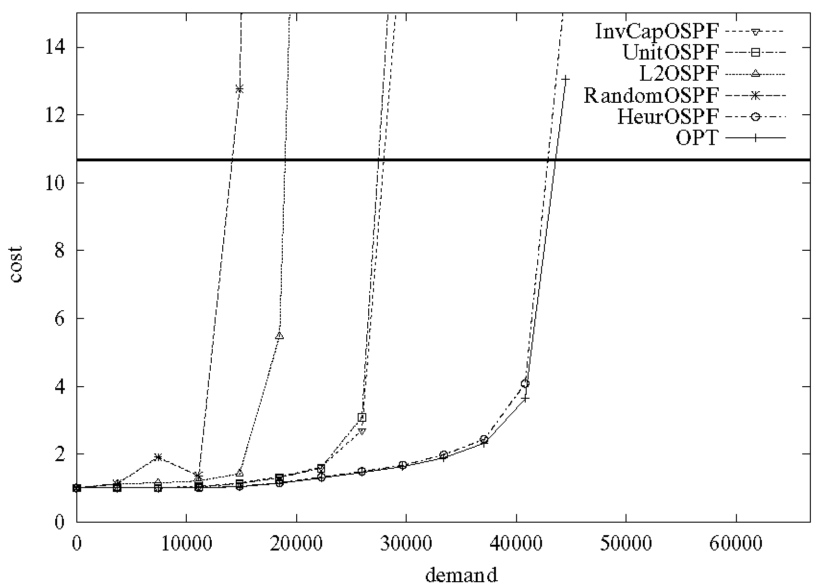
- use optimal multipliers as link weights
- non-zero flows only on shortest paths
- ECMP ≠ Optimal Flow Allocation
- good solution if most demand pairs only have one shortest path.

Heuristic Methods

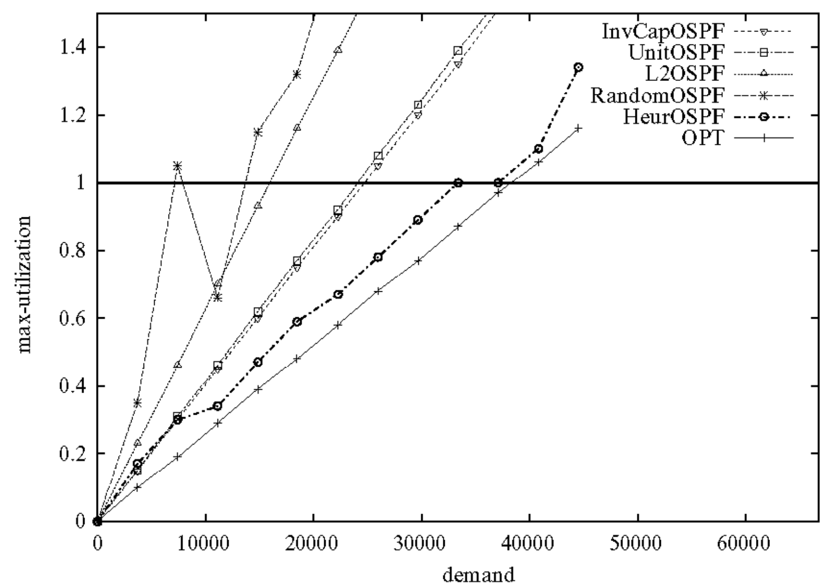
- Weight Adjustment
  - iterative local search
  - increase weights for over-loaded links, decrease weights for under-loaded links
  - adjust weights for more balanced allocation
- Simulated Annealing
  - random initial link weights
  - explore neighborhood: pick a random link, increase/decrease its weight by one
  - annealing: move to a worse weight setting with decreasing probability
- Lagrangian Relaxation (LR)-Based Dual Approach
  - optimum Lagrange multipliers lead to optimal solution
  - iterative algorithm to solve problem in dual space
  - Step1: given a set of multipliers, obtain link weights, and shortest path flow allocation
  - Step2: adjust multipliers according to link rates and link capacities, go back to step1 if stopping criteria not satisfied.

Example: impact of different link weight systems

- AT&T 90-node WorldNet IP Backbone
- scaled up demand volumes



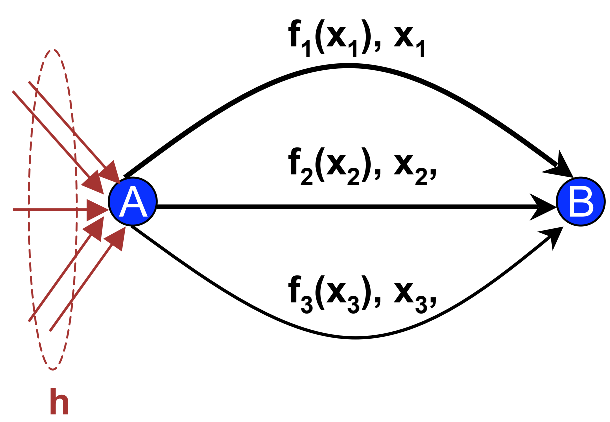
average delay



maximum link utilization

Extensions

- Un-capacitated Shortest-Path Routing
  - modularized dimensioning
- Optimizing link weights under transient failures
  - 50% network failures < 1 min; 80% < 10 min.s
  - no time to re-compute weights after each failure
  - good weight setting for both normal and failure situation
- Selfish Routing and Optimal Routing
  - every user choose minimum delay path
  - Nash Equilibrium vs. Social Optimum



Braess Paradox

- adding a link increase user delay