1. Let $X$ represent the lifetime of a certain electric bulb, and $Y$ that of its replacement after the failure of the first bulb. Suppose $X$ and $Y$ are independent random variables with exponential density function with common parameter $\lambda$. Find the probability that the combined lifetime exceeds $2\lambda$. What is the probability that the replacement outlasts the original component by $\lambda$?

2. Let

$$f_{XY}(x, y) = \begin{cases} 2e^{-(x+y)}, & 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Define $Z = \max(X, Y)$, $W = X/Y$. Determine the joint p.d.f of $Z$ and $W$. Are $Z$ and $W$ independent random variables?

3. $X$ and $Y$ are independent random variables with Geometric probability marginal functions

$$P(X = k) = pq^k, \quad k = 0, 1, 2, \ldots, \quad P(Y = m) = pq^m, \quad m = 0, 1, 2, \ldots.$$ 

Find the condition probability marginal function of $X$ given $X + Y$. (i.e., Determine $P(X = k | X + Y = n)$)

4. (a) The joint p.d.f of $X$ and $Y$ is given by

$$f_{XY}(x, y) = \begin{cases} 6x, & x > 0, \quad y > 0, \quad 0 < x + y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional p.d.f of $Y$ given $X$.

(b) $X$ and $Y$ are zero mean independent random variables with variances $\sigma_1^2$ and $\sigma_2^2$ respectively, i.e., $X \sim N(0, \sigma_1^2)$, $Y \sim N(0, \sigma_2^2)$. Let

$$Z = aX + bY + c, \quad c \neq 0.$$ 

Find the characteristic function $\Phi_Z(\omega)$ of $Z$. 

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