EL-630 Probability Theory

Final Exam: Spring 2008 (Answer all problems)

1. a). Let $X$ and $Y$ be zero mean, jointly normal identical random variables with correlation coefficient $\rho$. Define

$$Z = X \cos \theta + Y \sin \theta, \quad W = X \cos \theta - Y \sin \theta.$$ 

Find $\theta$ such that $Z$ and $W$ are independent.

b). Suppose $X$ and $Y$ are independent Poisson random variables with parameters $\lambda_1$ and $\lambda_2$ respectively. Determine the conditional probability mass function of $X$ given $X + Y$.

c). Suppose $X$ and $Y$ are identical random variables with correlation coefficient $\rho$. Are the variances of $X + Y$ and $X - Y$ equal? Justify.

2. The joint probability density function of $X$ and $Y$ is given by

$$f_{XY}(x, y) = \begin{cases} \frac{1}{2}(1-xy), & 0 < x < 1, \ -1 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$Z = \frac{X}{X + Y}.$$ 

Find the probability density function of $Z$.

3. Given the joint probability density function (p.d.f)

$$f_{XY}(x, y) = \begin{cases} xy e^{-(x+y)}, & x, y > 0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$Z = X + \max(X, Y), \quad W = \frac{X}{Y}.$$ 

Find the joint p.d.f. of $Z$ and $W$. Are $Z$ and $W$ independent?
4. $X$ and $Y$ are random variables with joint probability density function

$$f_{XY}(x, y) = \begin{cases} 2e^{-(x+y)}, & 0 < x < y < \infty, \\ 0, & \text{otherwise}. \end{cases}$$

Define

$$Z = X + Y, \quad W = X - Y.$$ 

(a) Find $f_{ZW}(z, w)$, the joint probability density function of $Z$ and $W$. 

(b) Are $Z$ and $W$ independent random variables? 

(c) Are $Z$ and $W$ uncorrelated random variables? 

5. The joint p.d.f. of $X$ and $Y$ is given by

$$f_{XY}(x, y) = \begin{cases} 6x, & x > 0, y > 1, 0 < x + y < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Find $E\{X^2 \mid Y = y\}$. 

(20)