EL-630 Solutions of HW #5

1. \( f_X(x) = \begin{cases} 
1/5, & -3 < x < 2 \\
0, & \text{otherwise.} 
\end{cases} \)

(a) \( P\{X^2 > 1\} = P\{-1 > X > 1\} = P\{X \in \text{shaded area above}\} \)

\[ = P\{(-3 \leq X < -1) \cup (1 < X \leq 2)\} \]

Disjoint events

\[ = P\{-3 \leq X < -1\} + P\{1 < X \leq 2\} = \int_{-3}^{-1} f_X(x)dx + \int_{1}^{2} f_X(x)dx \]

\[ = \int_{-3}^{-1} (1/5)dx + \int_{1}^{2} (1/5)dx = \frac{-1 - (-3)}{5} + \frac{2 - 1}{5} = \frac{2}{5} + \frac{1}{5} = \frac{3}{5} = 0.6 \]

(b) We need \( P\{\sin \pi X \geq 0\} \).

From figure \( \sin \pi X \geq 0 \) whenever

\(-2 \leq X \leq -1 \) and \( 0 \leq X \leq 1, \cdots \)

(Note: In general \( \sin \pi X \geq 0 \) whenever \( 2k \leq X \leq (2k + 1) \) for \( k = \cdots, -2, -1, 0, 1, 2, \cdots \)).

Hence
2. (a) we need $P(H)$. From the continuous version of Bayes’ Theorem, we have

$$P(H) = \int_0^1 P(H \mid X = p) f_X(x) \, dp$$  \hspace{1cm} (1)$$

But if $X$ is given to be $p$, then by different heads will occur with probability $p$. i.e.,

$$P(H \mid X = p) = p$$  \hspace{1cm} (2)$$

with (2) in (1)

$$P(H) = \int_0^1 p \, dp = \left. \frac{p^2}{2} \right|_0^1 = \frac{1}{2}$$  \hspace{1cm} (3)$$

(3) says, that in the absence of any other information the probability of a head occurring in the next toss is 0.5.

(b) Coin is tossed 10 times and heads show 6 times. Here $n = 10$, $k = 6$. Define this event to be $A$. i.e.,

$$A = \{k = 6 \; \text{in} \; n = 10\}$$
and (see lecture notes #4, Eq.(4-35))

\[
P\{A\} = \int_0^1 P\{A \mid X = p\} f_X(p) dp = \int_0^1 p^k q^{n-k} dp
\]

\[
= \frac{k!(n-k)!}{(n+1)!} = \frac{6!4!}{11!} = \frac{1}{2310}
\]

now

\[
P\{0.35 \leq X \leq 0.75 \mid A\} = \int_{0.35}^{0.75} f_{P,A}(p \mid A) dp.
\]

But from (4-36)

\[
f_{P,A}(p \mid A) = \frac{P\{A \mid p\} f_p(p)}{P\{A\}} = \frac{(n+1)!}{(n-k)!k!} p^k q^{n-k}, \quad 0 < p < 1.
\]

Which gives the required probability to be

\[
P\{0.35 \leq X \leq 0.75 \mid A\} = \frac{(n+1)!}{(n-k)!k!} \int_{0.35}^{0.75} p^k q^{n-k} dp \approx 0.835
\]

with \( n = 10, k = 6. \)
3. \( f_x(x) = \alpha e^{-\alpha x} U(x) \)

(a) Since \( X \) only taken positive values, so is \( Y \) and for every value of \( Y \), there is only one real solution for \( x = y^{1/3} \)

\[
\frac{dy}{dx} = 3x^2 = 3y^{2/3}
\]

\[
f_y(y) = \frac{1}{\frac{dy}{dx}} f_x(y^{1/3}) = \frac{1}{3y^{2/3}} \cdot \alpha e^{-\alpha y^{1/3}} U(y) = \begin{cases} 
\frac{\alpha}{3y^{2/3}} e^{-\alpha y^{1/3}}, & y \geq 0 \\
0, & \text{otherwise.}
\end{cases}
\]

\[
F_y(y) = P\{Y \leq y\} = P\{X^3 \leq y\} = P\{X \leq y^{1/3}\} = F_x(y^{1/3})
\]

\[
\Rightarrow f_y(y) = f_x(y^{1/3}) \cdot \frac{dy^{1/3}}{dy} = \frac{1}{3y^{2/3}} f_x(y^{1/3})
\]

(b) \( Y = 2X + 3 \)

\[
F_y(y) = P\{Y \leq y\} = P\{2X + 3 \leq y\} = P\{X \leq (y - 3) / 2\} = F_x((y - 3) / 2)
\]
or

\[
f_y(y) = \frac{d}{dy} F_x((y - 3) / 2) = f_x((y - 3) / 2) = \frac{1}{2} f_x((y - 3) / 2) = \frac{\alpha}{2} e^{-\alpha(y-3)/2}.
\]
4. \( P\{X = k\} = \binom{n}{k} p^k q^{n-k}, \ k = 0,1,2,\ldots,n \)

(a) \( Y = X^2 + 1 \). Since \( X \) is discrete, \( Y \) is also discrete. In fact when \( X = k \), \( Y \) takes the value \( k^2+1 \). So when \( X \) ranges over \( 0,1,2,\ldots,n \), \( Y \) ranges over \( 1,2,5,\ldots,k^2+1,\ldots,n^2+1 \)

\[
P\{Y = l\} = P\{X^2 + 1 = l\} = P\{X = \sqrt{l-1}\} = \binom{n}{\sqrt{l-1}} p^{\sqrt{l-1}} q^{n-\sqrt{l-1}}.
\]

Hence

\[
P\{Y = l\} = \binom{n}{\sqrt{l-1}} p^{\sqrt{l-1}} q^{n-\sqrt{l-1}}, \ l = 1,2,5,\ldots,n
\]

(b) \( Y = \sqrt{X} \). Again \( Y \) is discrete. Moreover when \( X \) ranges over \( 0,1,2,\ldots,n \), \( Y \) ranges over \( 0,1,2,\ldots,\sqrt{n} \), respectively

\[
P\{Y = l\} = P\{\sqrt{X} = l\} = P\{X = l^2\} = \binom{n}{l^2} p^{l^2} q^{n-l^2}, \ l = 0,1,2,3,\ldots,\sqrt{n}.
\]

(c) \( Y = e^{-X} \). Again \( Y \) is discrete. Moreover when \( X \) ranges over \( e^0,e^{-1},e^{-2},\ldots,e^{-n} \) and

\[
P\{Y = l\} = P\{e^{-X} = l\} = P\{X = -\log l\} = \binom{n}{-\log l} p^{-\log l} q^{n+\log l}, \ l = 1,e^{-1},e^{-2},\ldots,e^{-n}.
\]

5. (a) \( Y = \frac{X}{1+X} \Rightarrow (1+X)Y = X \) or \( X = \frac{Y}{1-Y} \). Note \( Y \) is a 1-1 mapping.

\[
F_Y(y) = P\{Y \leq y\} = P\{X/(1+X) \leq y\} = P\{X \leq y(1+X)\}
\]

\[
= P\{X \leq y/(1-y)\} = F_X(y/(1-y)).
\]
and
\[ f_Y(y) = f_X(y/(1-y)) \frac{d}{dy}\left(\frac{y}{1-y}\right) = f_X(y/(1-y)) \cdot \left(\frac{(1-y)+y}{(1-y)^2}\right) = \frac{1}{(1-y)^2} f_X(y/(1-y)). \]

If \( X \sim U(0,1) \)
\[ f_Y(y) = \frac{1}{(1-y)^2}, \quad 0 \leq y \leq \frac{1}{2}. \]

(b) \( X \sim N(0, \sigma^2) \). i.e., \( f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}, \quad -\infty \leq x \leq +\infty \) \hspace{1cm} (1)

i) \( Y = X^2 \); Two solutions \( X_1 = \sqrt{y}, \ X_2 = -\sqrt{y} \)
\[ F_Y(y) = P\{Y \leq y\} = P\{X^2 \leq y\} = P\{-\sqrt{y} \leq X \leq \sqrt{y}\} = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \]
\[ f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) = \frac{1}{2\sqrt{y}} [f_X(\sqrt{y}) + f_X(-\sqrt{y})] \] \hspace{1cm} (2)

or with (1) in (2)
\[ f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-y/2\sigma^2}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases} \]
ii) $Z = |X|$

$$F_z(z) = P\{Z \leq z\} = P\{|X| \leq z\} = P\{-z < X \leq z\} = F_x(z) - F_x(-z)$$

$$f_z(z) = f_x(z) + f_x(-z) \hspace{1cm} (3)$$

with (1) in (3)

$$f_z(z) = \begin{cases} \sqrt{\frac{2}{\pi \sigma^2}} e^{-z^2/2\sigma^2}, & z \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

6.

$$f_x(x) = \begin{cases} 0, & x \leq 0 \\ 1/2, & 0 < x \leq 1 \\ 1/(2x^2), & x > 1 \end{cases}$$

a) $Z = \frac{1}{X}$. Since $f_x(x) = 0$ for $x < 0 \Rightarrow f_z(z) = 0$ for all $z < 0$. (i.e., probability of $Z$ being

$$F_z(z) = P\{Z \leq z\} = P\{1/X \leq z\} = P\{X \geq 1/z\} = 1 - F_x(1/z)$$

$$f_z(z) = -F_x(1/z) \cdot \frac{d(1/z)}{dz} = \frac{1}{z^2} f_x(1/z)$$
Note that $X$ and $Z$ have identical p.d.f. $X$ taken negative values with zero probability and since $Z = 1/X$, $Z$ also taken negative values with zero probability, or $f_Z(z) = 0$, for $Z \leq 0$.

b) $Y = \sec X$

From $Y = \sec X$ graph, note that $Y$ doesn’t take any values in the interval $|Y| < 1$,

$$f_Y(y) = 0 \quad \text{for } |Y| < 1.$$
For all other \( y \), there are two values of \( x \) that satisfy \( Y = \sec X \) (i.e., \( \sec X_1 = \sec X_2 = Y \)). Hence

\[
f_y(y) = \frac{1}{dy} f_x(x_1) + \frac{1}{dy} f_x(x_2)
\]

Moreover

\[
f_x(x_i) = \frac{1}{2\pi}, \quad f_x(x_2) = \frac{1}{2\pi}.
\]

Hence

\[
f_y(y) = \frac{1}{y\sqrt{y^2 - 1}} \left[ \frac{1}{2\pi} + \frac{1}{2\pi} \right], \quad |y| \geq 1.
\]

Then

\[
f_y(y) = \begin{cases} 
\frac{1}{\pi y\sqrt{y^2 - 1}}, & |y| \geq 1 \\
0, & otherwise.
\end{cases}
\]

7. a) \( M = (X \leq 0) \cup (1 < X \leq 2) \).

If \( x \leq 0 \)

\[
F_x(x \mid M) = P\{X \leq x \mid M\} = \frac{P\{(X \leq x) \cap M\}}{P\{M\}}
\]
\[ F_x(x \mid M) = \frac{P\{(X \leq x) \cap [(X \leq 0) \cup (1 < X \leq 2)]\}}{P\{M\}} \]

\[ = \frac{P\{(X \leq x) \cap (X \leq 0)\}}{P\{M\}} = \frac{P\{X \leq x\}}{P\{M\}} = \frac{F_x(x)}{P\{M\}} \quad x \leq 0. \]  

\[ \text{If } 0 < x \leq 1 \]

\[ F_x(x \mid M) = P\{X \leq x \mid M\} = \frac{P\{(X \leq x) \cap M\}}{P\{M\}} \]

\[ = \frac{P\{(X \leq x) \cap [(X \leq 0) \cup (1 < X \leq 2)]\}}{P\{M\}} = \frac{P\{X \leq 0\}}{P\{M\}} = \frac{F_x(0)}{P\{M\}}. \]  

\[ \text{If } 1 < x \leq 2 \]

\[ F_x(x \mid M) = \frac{P\{(X \leq x) \cap [(X \leq 0) \cup (1 < X \leq 2)]\}}{P\{M\}} \]

\[ = \frac{P\{[(X \leq x) \cap (X \leq 0)] \cup [(X \leq x) \cap (1 < X \leq 2)]\}}{P\{M\}} = \frac{P\{X \leq 0\} \cup (1 < X \leq x)}{P\{M\}} \]

\[ = \frac{P\{X \leq 0\} + P\{1 < X \leq x\}}{P\{M\}} = \frac{F_x(0) + F_x(x) - F_x(1)}{P\{M\}}. \quad 0 < x \leq 1 \]  

\[ \text{If } x > 2 \]

\[ F_x(x \mid M) = P\{X \leq x \mid M\} = \frac{P\{(X \leq x) \cap M\}}{P\{M\}} \]

\[ = \frac{P\{(X \leq x) \cap [(X \leq 0) \cup (X \leq 2)]\}}{P\{M\}} = \frac{P\{M\}}{P\{M\}} = 1 \quad x > 2 \]
where

\[ P\{M\} = P\{(X \leq 0) \cup (1 < X \leq 2)\} = P\{X \leq 0\} + P\{1 < X \leq 2\} = F_x(0) + F_x(2) - F_x(1). \]

From (1), (2), (3) and (4)

\[ F_x(x \mid M) = \begin{cases} 
F_x(x) / P\{M\}, & x \leq 0 \\
F_x(0) / P\{M\}, & 0 < x \leq 1 \\
(F_x(0) - F_x(1) + F_x(x)) / P\{M\}, & 1 < x \leq 2 
\end{cases} \]

and from the above,

\[ f_x(x \mid M) = \frac{d(F_x(x \mid M))}{dx} = \begin{cases} 
f_x(x) / P\{M\}, & x \leq 0 \\
0, & 0 < x \leq 1 \\
f_x(x) / P\{M\}, & 0 < x \leq 1, 1 < x \leq 2 \\
0, & x > 2 
\end{cases} \]

or

\[ f_x(x) = \begin{cases} 
f_x(x) / P\{M\}, & (x \leq 0) \cup (1 < x \leq 2) \\
0, & otherwise. 
\end{cases} \]

In all these cases

\[ P\{M\} = \{F_x(0) + F_x(2) - F_x(1)\}. \]
b) $Y = \sin X$ has infinite solutions $y = \sin x_1 = \sin x_2 = \cdots$. But the p.d.f of $X$ is nonzero only in $(0, \pi)$ and in that region $y = \sin x$ has two solutions $x_1 = \sin y$, $x_2 = \pi - \sin^{-1}(y)$.

$$
\frac{dy}{dx} = \cos x = \sqrt{1 - \sin^2 x}
$$

$$
\left| \frac{dy}{dx}_1 \right| = \sqrt{1 - \sin^2 x_1} = \sqrt{1 - y^2},
$$

$$
\left| \frac{dy}{dx}_2 \right| = \sqrt{1 - \sin^2 x_2} = \sqrt{1 - y^2}
$$

$$
f_x(x) = \begin{cases} 
2x & , 0 < x < \pi \\
\frac{2}{\pi^2}, & otherwise.
\end{cases}
$$

$$
f_y(y) = \sum \frac{1}{|dy/\text{dx}|} f_x(x) = \frac{1}{\sqrt{1 - y^2}} \{f_x(x_1) + f_x(x_2)\}
$$

$$
= \frac{1}{\sqrt{1 - y^2}} \{2/\pi^2[\sin^{-1}(y) + \pi - \sin^{-1}(y)]\}
$$

$$
= \frac{2}{\pi^2 \sqrt{1 - y^2}} \cdot \pi = \frac{2}{\pi} \frac{1}{\sqrt{1 - y^2}}, \quad 0 < y < 1.
$$