1. Here the combined experiment $\Omega = \Omega_1 \times \Omega_2 \times \Omega_3 \times \Omega_4$ and elementary event $\xi \in \Omega$ is of the form $\xi = \xi_1 \xi_2 \xi_3 \xi_4$, where $\xi_i$ is the face value $f_i$ (1 for $H$) and (0 for $T$) of the $i^{th}$ toss. The random variable $X$ is defined as

$$X = X(\xi) = \sum_{i=1}^{4} f_i$$

where $f_i = 1$ if $\xi_i = H$ (as $i^{th}$ toss) and $f_i = 0$ if $\xi_i = T$.

$\{X = 3\} = \{\xi \in \Omega \mid X(\xi) = 3\} = \{HHHT, HHTH, HTHH, THHH\}$

$\{X \leq 2\} = \{TTTH, TTHT, THTT, HTTT, TTTT, TTHH, THHT, HHTT, THTH, HTHT, HTTH\}$

$\{X < 2\} = \{TTTH, TTHT, THTT, HTTT, TTTT\} \subset \{X \leq 2\}$

$\{X = 5\} = \phi$.

2. A distribution function must be monotonically non-decreasing right continuous, and non-negative on the entire real axis with $F(-\infty) = 0$, $F(+\infty) = 1$.

(i) $F_X(x)$ has solution all the above requirements (verify!)

a. $\begin{cases} F(-\infty) = 0, \quad \text{(given)} \\ F(+\infty) = 1 - e^{-\alpha x} = 1 - 0 = 1 \end{cases}$

b. It is non-negative as the entire real axis.

c. Right continuity needs to be verified only at $x = 0$ (here) as the function is given to be continuous everywhere else at $x = 0$, $F_X(0) = 0$, $F_X(0^+) = \lim_{\epsilon \to 0}(1 - e^{-\alpha \epsilon}) = 1 - 1 = 0$
\( F_X(x) \) is right continuous on the entire real axis.

\( d. \) It is non-decreasing everywhere. \( F_X(x) \) is a valid P.D.F.

(ii)

\[
F_X(x) = \begin{cases} 
0, & x < 0 \\
x, & 0 \leq x \leq \frac{1}{2} \\
1, & x > \frac{1}{2} 
\end{cases} \quad (1)
\]

Note that from (2) \( F_X(1/2) = 1/2 \). However from (3), \( F_X(1/2^+) = 1 \) (right limit at \( x = 1/2 \))

\[
F_X(x) \neq F_X(x^+) \text{ at } x = 1/2
\]

and hence it is not a valid P.D.F.

(iii) Here \( F_X(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x), \ -\infty < x < \infty \).

\[
F_X(-\infty) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(-\infty) = \frac{1}{2} + \frac{1}{\pi} (-\frac{\pi}{2}) = 0, \quad F_X(+\infty) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(+\infty) = \frac{1}{2} + \frac{1}{\pi} (\frac{\pi}{2}) = 1.
\]

The function \( F_X(x) \) is continuity everywhere (and hence continuous from right). It is also non-negative and monotonically non-decreasing (follows from \( \tan(\cdot) \) curve).

\[
F_X(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x), \ -\infty < x < \infty \text{ is a valid P.D.F}
\]
3. (i) If \( F_X(x) \) is a valid p.d.f, then \( f_X(x) \geq 0 \) everywhere and \( \int_{-\infty}^{+\infty} f_X(x) dx = 1 \). Here 
\[
\int_{-\infty}^{+\infty} f_X(x) dx = \int_{-5}^{+5} (1 + \sin^{10}(kx)) dx > \int_{-5}^{+5} 1 dx = 10.
\]
Thus \( \int_{-\infty}^{+\infty} f_X(x) dx > 10 \) and hence \( f_X(x) \) can not represent a valid p.d.f.

(ii) \( P[x = k] = k p^k; \ k = 0, 1, 2, \ldots; \ 0 < p < 1 \). Note that \( P[x = k] > 0 \), further for discrete random variables we require
\[
\sum_{k=0}^{\infty} P[x = k] = 1
\]
But
\[
\sum_{k=-\infty}^{\infty} P[x = k] = \sum_{k=0}^{\infty} k p^k = k \sum_{k=0}^{\infty} p^k = k(1 + p + p^2 + \cdots) = \frac{k}{1 - p} = 1
\]
\[
\Rightarrow k = 1 - p = q
\]
or \( P[x = k] = q p^k \) is a valid probability mass function (p.m.f).

(iii) \( f_X(x) = \begin{cases} \frac{k}{x} e^{-(\ln x - \mu)^2/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \)

Note that \( f_X(x) \geq 0 \) everywhere. Now we need \( k \) such that
\[
\int_{0}^{+\infty} \frac{k}{x} e^{(\ln x - \mu)^2/2} dx = 1
\]
substitute \( \ln x = y \Rightarrow \frac{dx}{x} = dy \) and when \( x = 0, \ y = -\infty \) and when \( x = \infty, \ y = \infty \)
\[ \int_0^\infty k e^{-(\ln x - \mu)^2/2} \frac{dx}{x} = \int_0^\infty k e^{-(\ln x - \mu)^2/2} dy = k \int_{-\infty}^{+\infty} e^{-(y - \mu)^2/2} dy \]

Further let \( y - \mu = z \). Then \( y = -\infty \Rightarrow z = -\infty \) and \( y = \infty \Rightarrow z = \infty \) and \( dy = dz \)

\[ k \int_{-\infty}^{+\infty} e^{-(y - \mu)^2/2} dy = k \int_{-\infty}^{+\infty} e^{-z^2/2} dz = 2k \int_0^{+\infty} e^{-z^2/2} dz \]

as \( e^{-z^2/2} \) is an even function of \( z \). Note, \( \int_0^{+\infty} e^{-z^2/2} dz = \sqrt{\pi}/2 \) (see Gamma functions)

\[ 2k \int_0^{+\infty} e^{-z^2/2} dz = 2k \sqrt{\pi}/2 = 1 \] or \( k = \frac{1}{\sqrt{2\pi}} \).

Thus

\[ f_x(x) = \frac{1}{x \sqrt{2\pi}} e^{-(\ln x - \mu)^2/2}, \quad 0 < x < \infty \]

is a valid p.d.f for any \( \mu \).

4. From past data, fatality rate \( p = 1/1000 = 0.001 \) person/thousand. Let \( x \) represent the number of people (among 10,000) that will die of horse kick. Then \( x \) is given to be a Poisson random variable and its parameter \( \lambda_0 \) for a 10,000 crowd is

\[ \lambda_0 = np = 1000 \times \frac{1}{1000} = 10 \]

we need

\[ P[x \leq 3] = \sum_{k=0}^{3} P[x = k] \]

but

\[ P[x = k] = e^{-\lambda_0} \frac{\lambda_0^k}{k!} \]
Hence

\[ P[x \leq 3] = e^{-\lambda} \sum_{k=0}^{3} \frac{\lambda^k}{k!} = e^{-10} \left[ 1 + \frac{(10)^1}{1!} + \frac{(10)^2}{2!} + \frac{(10)^3}{3!} \right] \]

\[ = e^{-10} [1 + 10 + 50 + 187] = e^{-10} \times 228 \approx 0.01035 \]

5. The random variables \( X \) and \( Y \) are such that \( X(\xi) \leq Y(\xi) \) for every \( \xi \). Thus for example

![Diagram of random variables X and Y]

Hence if \( Y(\xi) \leq w_1 \) \( \Rightarrow \) \( X(\xi) \leq w_1 \). But if \( X(\xi) \leq w_2 \), this does not imply \( \not\Rightarrow \) \( Y(\xi) \leq w_2 \) (see figure).

\[ \{Y(\xi) \leq w\} \subseteq \{X(\xi) \leq w\} \text{ for any } w \]

Hence \( P[Y(\xi) \leq w] \leq P[X(\xi) \leq w] \) or \( F_y(w) \leq F_x(w) \).
6. Refer to (3-54)-(3-55) Text. When stakes are doubled \((k = 2)\), from (3-55)

\[
P_a = P_a^* \cdot \frac{1 + (p/q)^{b/2}}{1 + (p/q)^{(a+b)/2}} = P_a^* \cdot c
\]

or \(P_a^* = P_a / c\). Here \(P_a\) is given by (3-47). Note that \(c > 1\) when \(p < q\) (Why?). Here \(P_a^* < P_a\) when \(p < q\). Thus when playing a series of unfavorable games, it is better to maintain high stakes at each game.

![Fig. 1](a = 20, b = 2)

![Fig. 2](a = 100, b = 10)

![Fig. 3](a = 5, b = 2)
7. Use Eq (3-47) in Text with \( a = m, b = n \), for \( p = 0.3, p = 0.45 \) and \( p = 0.49 \). The probability of ruin \( P_a \) is plotted below for \( m = 100 \). From Fig.1 for a $100 investment, a 10% return \( (n = 10) \) can be expected with probability 0.13 (since \( P_a = 0.87 \)) for games played with \( p = 0.45 \). On the other hands if \( p = 0.49 \), a 10% return can be expected with probability 0.68, and a 60% return can be expected with probability 0.1.

Thus improving the skills from \( p = 0.4 \) to \( p = 0.45 \) increases the probability of return from 0.01 to 0.13 for a $10 return. On the other hand increasing the skills from \( p = 0.45 \) to \( p = 0.49 \) increases the probability of return from 0.13 to 0.68, in a disproportionate manner.

![Graph showing probability of ruin for different probabilities of return](image)

Thus, small differences in skills at the higher end can lead to dissimilar returns.