EL630 - Solutions to HW #1

1. Note that $A, B$, mutually exclusive implies $A \cap B = \emptyset$ and hence $P(A \cap B) = \emptyset$. Further $A, B$ are independent $\Rightarrow P(A \cap B) = P(A)P(B)$

Let the pair $(f_i, s_k)$ represent the elementary event “outcome on first roll is $i$ and second roll is $k$” $i, k = 1, 2, \cdots, 6$. We have 36 such elementary events.

a) $A =$ "first outcome is odd"

$= \{(f_1, s_1), \cdots, (f_1, s_6), (f_3, s_1), \cdots, (f_3, s_6), (f_5, s_1), \cdots, (f_5, s_6)\}$

$B =$ "first outcome is even"

$= \{(f_2, s_1), \cdots, (f_2, s_6), (f_4, s_1), \cdots, (f_4, s_6), (f_6, s_1), \cdots, (f_6, s_6)\}$

Clearly $A \cap B = \emptyset \Rightarrow P(A \cap B) = \emptyset$. But $P(A) = 1/2, P(B) = 1/2$ (why?) $\Rightarrow P(A \cap B) \neq P(A)P(A) \Rightarrow$ Through $A, B$ are Mutually Exclusive (M.E), they are not independent.

b) $A =$ “first outcome is 1” $= \{(f_1, s_1), (f_1, s_2), \cdots, (f_1, s_6)\}$

$B =$ “second outcome is 2” $= \{(f_1, s_2), (f_2, s_2), \cdots, (f_6, s_2)\}$. Thus $A \cap B = \{(f_1, s_2)\} \neq \emptyset$. $\Rightarrow A, B$ are not M.E.

But $P(A \cap B) = P[(f_1, s_2)] = P[outcome on first roll is 1. and that on second roll is 2]$ (Since, and belong to distinct trials, they do not influence each after, or $A, B$ are independent)

$P(A \cap B) = P[ outcome of 1^{st} roll is 1] \cdot P[ outcome on 2^{nd} roll is 2]$

$= P(A)P(B) \Rightarrow A, B$ are independent.
c) \( A = "\text{first outcome is 7}" = \phi \)
\( B = "\text{second outcome is odd}" \)

\( A \cap B = \phi \) and also \( A \) and \( B \) are independent since they belong to separate trials.

**Remark:** Independence and mutually exclusive together implies

\[
P(A)P(B) = P(A \cap B) = P(\phi) = 0
\]

Thus either \( P(A) = \phi \) or \( P(B) = 0 \). (Here \( P(A) = 0 \).

\[d) \quad A = \text{"first outcome is odd"} = \{(f_1, s_1), (f_1, s_2), \ldots, (f_3, s_1), \ldots\}
\]
\[B = \text{"first outcome is one"} = \{(f_1, s_1), (f_1, s_2), \ldots, (f_1, s_6)\}.
\]

Thus
\[
A \cap B = \{(f_1, s_1), \ldots, (f_1, s_6)\} \neq \phi.
\]

Also \( A, B \) are not independent as they belong to the same roll.

2. \( P(A) = 1/4, \ P(B) = 2/5 \)

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)}
\]

But \( A \subset B \quad \Rightarrow \quad A \cap B = A \)

Thus
\[
P(A \mid B) = \frac{P(A)}{P(B)} = \frac{1/4}{2/5} = 5/8
\]
and

\[ P(B \mid A) = \frac{P(A \cap B)}{P(A)} - \frac{P(A)}{P(A)} = 1. \]

3. \(A, B\) form a partition for the first trial. (why?), because \(A \cap B = \emptyset\) and \(A \cup B =\) certain event, \(P(A \cup B) = 1\). Similarly \(C, D\) forms a partition for the second trial. Thus from the over of total probability

\[ P(A) = P(A \mid C)P(C) + P(A \mid D)P(D) = P(A, C) + P(A, D) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \]

and similarly

\[ P(C) = P(C \mid A)P(A) + P(C \mid B)P(B) = P(C, A) + P(C, B) \]

\[ = P(A, C) + P(B, C) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}. \]

Thus

\[ P(A)P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}. \]

But

\[ P(A, C) = P(A \cap B) = P(AC) = \frac{1}{3}. \]

\(A\) and \(C\) are not independent (because \(P(A \cap B) \neq P(A)P(B)\)).
4. Let \( C_1 \) and \( C_2 \) represent the first and second child respectively. \( C_1 \) can be a boy (\( B \)) or a girl (\( G \)) with equal probability, same is true with \( C_2 \). Thus

\[
P(C_1 = B) = P(C_1 = G) = \frac{1}{2}
\]

and

\[
P(C_2 = B) = P(C_2 = G) = \frac{1}{2}
\]

Further the two events corresponding to the birth of first and second child are independent. i.e.,

\[
P(C_1 = X, C_2 = Y) = P(C_1 = X)P(C_2 = Y)
\]

where \( X, Y \) can be \( B \) or \( G \).

a) \( P(\text{Both children are boys} \mid \text{the older child is a boy}) = P[(C_1 = B, C_2 = B) \mid (C_1 = B)] \)

\[
= \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{P[(C_1 = B, C_2 = B)]}{P(C_1 = B)}
\]

\[
= \frac{P(C_1 = B) \cdot P(C_2 = B)}{P(C_1 = B)} = P(C_2 = B) = \frac{1}{2}.
\]

b) \( P(\text{Both children are boys} \mid \text{at least one is a boy}) = P(A \mid C) \)

\( C = "\text{at least one child is a boy}" = "\text{one child is a boy and the other is a girl}" \)

\( \cup "\text{Both children are boys}"
\)

\[
= [(C_1 = B, C_2 = G) \cup (C_1 = G, C_2 = B)] \cup [C_1 = B, C_2 = B]
\]
\[ P[C] = P\{(C_1 = B, C_2 = G) \cup (C_1 = G, C_2 = B) \cup (C_1 = B, C_2 = B)\} \\
= P(C_1 = B, C_2 = G) + P(C_1 = G, C_2 = B) + P(C_1 = B, C_2 = B) \\
= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}. \]

Thus
\[
P(A \mid C) = \frac{P(A \cap B)}{P(C)} = \frac{P(A)}{P(C)} = \frac{P(C_1 = B, C_2 = B)}{P(C)} \\
(\text{Note } A \subset C \text{ and hence } A \cap C = A) \\
= \frac{P(C_1 = B)P(C_2 = B)}{P(C)} = \frac{(1/2)(1/2)}{(3/4)} = \frac{1}{3}. \]

5. Let \( X_i \) denote the event that there are exactly \( i \) defective bulbs in the box; evidently \( i = 0, 1, 2, \ldots, 1000 \). So that elementary events are \( X_0, X_1, X_2, \ldots, X_{1000} \) and hence
\[
\Omega = \{X_0, X_1X_2, \ldots, X_{100}\} \\
A = "\text{there is at least one defective bulb}" \\
= "\text{there are one or more defective bulbs}" \\
= X_1 \cup X_2 \cup X_3 \cup \cdots \cup X_{100}. \]

Here \( \overline{A} = X_0 = "\text{there are no defective bulbs}" \)
a) \( P[\text{No defective bulb}] = P(\bar{A}) = 1 - P(A) = 1 - 0.1 = 0.9. \)

\[ B = \text{"there are at least two defective bulbs"} = X_2 \cup X_3 \cup \cdots \cup X_{1000} \]

and

\[ \bar{B} = X_0 \cup X_1 = \text{"there is at most one defective bulb"} \]

and

\[ A = X_1 \cup X_2 \cup \cdots \cup X_{1000}. \]

Thus

\[ A \cap \bar{B} = X_1 = \text{"there is exactly one defective bulb"} \]

always

\[ A = A(B + \bar{B}) = A \cap B + A \cap \bar{B} \]

or

\[ P(A) = P(A \cap B) + P(A \cap \bar{B}) \] as \( A \cap B \) and \( A \cap \bar{B} \) are M.E

\[ \Rightarrow \quad P(A \cap \bar{B}) = P(A) - P(A \cap B). \]

Here \( B \subset A \quad \Rightarrow \quad A \cap B = B \)

\[ P(X_1) = P(A \cap \bar{B}) = P(A) - P(B) = 0.1 - 0.04 = 0.06 \]

c) \( P(\text{"at least one defective bulb")} = P(\bar{B}) = 1 - P(B) = 1 - 0.04 = 0.96. \)

6. Let \( M = \text{Man}, \ W = \text{Women}, \ B = \text{Blond}. \) Among the people 60% are men. If you randomly pick any one, the probability of that person being a man is 0.6
or \( P(M) = 0.6 \) and \( P(W) = 0.4 \). Among the men 30\% are blond. If you randomly pick any 
man, then the probability of that man \( (M) \) being blond \( (B) \) is 30\%.

\[
P[\text{Being Blond given that a man is picked}] = P(B | M) = 0.3.
\]

Similarly

\[
P(B | W) = 0.4 \text{ (given)} \quad \text{(If you pick any woman at random with 40\% probability. She will be blond)}
\]

First we compute

\[
P(B) = P(\text{A randomly chosen person being blond})
\]

\[
= P(B | M)P(M) + P(B | W)P(W) = 0.3 \times 0.6 + 0.4 \times 0.4 = 0.18 + 0.16 = 0.34.
\]

We pick a person at random and is found to be blond. Now that person can be \( M \) or \( W \). we need the probability of that person being a man. i.e.,

\[
P\left( \text{Picked person is Man given that the person is Blond} \right) = \frac{P(B | M)P(M)}{P(B)} = \frac{0.3 \times 0.6}{0.34} = \frac{9}{17}.
\]

Read

1. Chapter #2, 3.1, 3.2, 3.4 in Papoulis [1].
2. Section 1.2, 1.3, 1.4, 1.5, 1.6 in [2].
3. You may also refer the book.