On the Effect of Imperfect Interleaving for the Gilbert–Elliott Channel

Leif Wilhelmsson, *Member, IEEE,* and Laurence B. Milstein, *Fellow, IEEE*

***Abstract*— By using the Gilbert–Elliott model to study the performance of block-coded transmission over the land mobile channel, a new analytical expression illustrating the effect of various parameters, e.g., mobile speed, delay constraint, and parameters for the error correcting code, is found. Comparisons between the results obtained by this analytical expression and results obtained by computer simulations show that the analytical results are accurate for a broad range of channel parameters. The Gilbert–Elliott model is then used to compare the performance of different binary BCH codes when the delay constraint does not allow the assumption of infinite interleaving. In contrast to the memoryless case, where the performance typically is improved with increased block length, short codes are found to be as good, or even superior, due to the fact that the interleaver works better for shorter codes.**

***Index Terms*—Block coding, Gilbert–Elliott channel, imperfect interleaving, land mobile channel, Rayleigh fading.**

I. INTRODUCTION

STUDIES of the performance of error correcting codes are most often concerned with situations where the channel is assumed to be memoryless, since this allows for a theoretical analysis. In situations where memory is accounted for, on the other hand, the analytical results are few, and studies of the performance are therefore often obtained via simulations.

To deal with a complicated channel model, it is sometimes possible to use a less complex one that still reflects the essential (for that particular study) properties of the complicated model. For a channel with memory, the Gilbert–Elliott (GE) channel, which emerges from the early 1960's and is due to Gilbert [1] and Elliott [2], is one of the simplest models. In this model for a slowly varying channel, the channel is assumed to either be in a good state, where the probability of error is small, or in a bad state, where the probability of error is significantly larger. The dynamics of the channel are modeled as a first-order Markov chain, a model which Wang and Moayeri [3] and Wang and Chang [4], in spite of its simplicity, showed to be very accurate for a Rayleigh fading channel. In [5], Ahlin presented a way to match the parameters of the GE model to the land mobile channel, an approach that was generalized in [3] to a Markov model with more than two states. This approach was also used by Sharma *et al.* in [6] where error trapping decoders are studied.

In this work, we use the GE channel to evaluate the performance of random error correcting block codes for the land mobile channel, e.g., [7]. First, for block-coded transmission over the GE channel, we derive the exact probability of error as a function of the channel parameters, the interleaver parameters, and the code. This probability of error is easy to obtain numerically, and hence no lengthy simulations are needed. A related problem has also been considered by Yee and Weldon [8], although the derivations of the desired formulas are done somewhat differently in their work. Then, using the approach suggested by Ahlin, the parameters of the GE channel are matched to the land mobile channel. Since the GE channel is a very simple model, it is not obvious that it will give accurate results just because some of the channel's parameters are matched. Here we confirm that it is possible to obtain accurate results by using the GE model, and that small variations of the GE model's parameters do not significantly affect the accuracy. Finally, by using the GE model, codes of different complexity are compared when used in a situation where the interleaver cannot be made sufficiently large to work appropriately because of delay constraints reasonable for voice communication. These comparisons show that codes of high complexity can be outperformed by a simpler code, because the interleaver works better for the latter class of codes.

The remainder of the paper is organized as follows. In Section II, notations are introduced, some known properties of the channel model are recapitulated, and useful statistical properties of the channel are derived. Section III describes how the GE model can be matched to the land mobile channel. An example is provided in order to show both that choosing the parameters for the GE model is relatively easy and that the results obtained by employing the GE model can be very accurate. In Section IV, comparisons between different codes are made to indicate how the interleaving process works for codes of varying complexity. Finally, in Section V, conclusions are drawn.

II. ANALYSIS OF BLOCK-CODED TRANSMISSION OVER AN INTERLEAVED GILBERT–ELLIOTT CHANNEL

The GE channel is a first-order, discrete-time, stationary, Markov chain with two states, one good and one bad, appropriately denoted G and B. In order to describe the channel, the following notation is used: The probabilities that the channel state changes from G to B and from B to G are denoted by b and g, respectively (see Fig. 1). The probabilities that

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L. Wilhelmsson is with Advanced Studies Research and Wideband Terminals, Ericsson Mobile Communications AB, SE-221 83 Lund, Sweden.

L. B. Milstein is with the Department of Electrical and Computer Engineering, University of California at San Diego, La Jolla, CA 92093-0407 USA.

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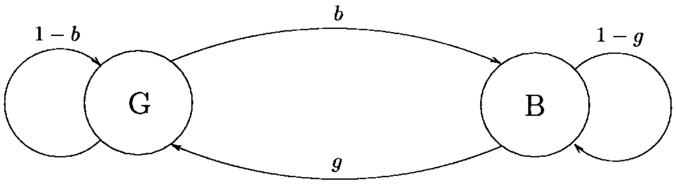

Fig. 1. The Gilbert–Elliott channel model.

the channel is in the good and bad state at the kth instant of time are denoted by $P^k(G)$ and $P^k(B)$, respectively, which, when matrix notation is adopted, will be written $\mathbf{P}^k = [P^k(G), P^k(B)]$. As a general rule, bold-faced typesetting is used for matrices and vectors. The probability of being in state G at time k, given that the channel is in state B at time 0, will be denoted $P^k(G|B)$, and the other possible conditional probabilities are denoted accordingly.

We let $\mathbf{T}$ denote the transition matrix for the channel, i.e.,

$$\mathbf{T} = \begin{bmatrix} 1-b & b \\ g & 1-g \end{bmatrix} \tag{1}$$

so that

$$\mathbf{P}^{k+1} = \mathbf{P}^k \mathbf{T}. \tag{2}$$

From (1) and (2), it is readily seen that how fast the channel is changing from one state to the other depends on b and g. The larger values of these parameters, the faster the channel is changing. For the channels that we are interested in, the channel is slowly changing compared to the symbol rate, and typically $b + g \ll 1$.

The stationary distribution is denoted $\mathbf{P}^\infty = [P^\infty(G), P^\infty(B)]$ and is easily found to be

$$\mathbf{P}^\infty = [g/(b+g),\, b/(b+g)]. \tag{3}$$

The average number of time units the channel spends in the good and the bad states are denoted $\overline{T}(G)$ and $\overline{T}(B)$, respectively. We immediately have

$$\begin{aligned} \overline{T}(G) &= b + 2(1-b)b + 3(1-b)^2 b + \cdots \\ &= b \sum_{k=1}^{\infty} k(1-b)^{(k-1)} = \frac{1}{b} \end{aligned} \tag{4}$$

and, analogously,

$$\begin{aligned} \overline{T}(B) &= g + 2(1-g)g + 3(1-g)^2 g + \cdots \\ &= g \sum_{k=1}^{\infty} k(1-g)^{(k-1)} \\ &= \frac{1}{g}. \end{aligned} \tag{5}$$

Finally, the probabilities of error for the good and bad states are denoted by $P_e(G)$ and $P_e(B)$, respectively.

To find the codeword error probability for an interleaved GE channel, we start by finding the effect an interleaver has, and then give an explicit expression for being in state B exactly d times out of n as a function of the parameters of the channel and the interleaver.

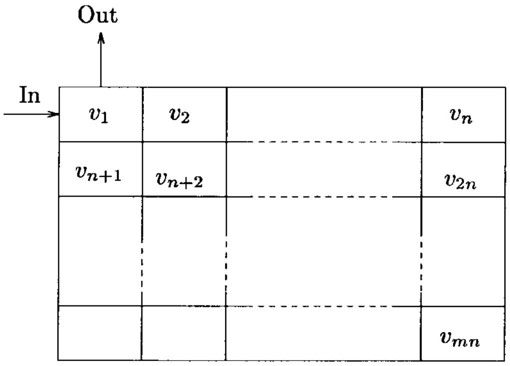

Fig. 2. The structure of the interleaver.

In referring to (2), it is clear that $\mathbf{P}^m = \mathbf{P}^0 \mathbf{T}^m$, from which it is easy to show that if the channel at time $t = 0$ is in the good state, the probabilities of being in the good and bad state at time m, respectively, are given by

$$\begin{aligned} \mathbf{P}^m &= [1,\, 0]\mathbf{T}^m \\ &= [P^\infty(G) + P^\infty(B)(1-b-g)^m, \\ &\qquad P^\infty(B)(1-(1-b-g)^m)] \end{aligned} \tag{6}$$

where we have used the stationary probability distribution (3). Similarly, if $\mathbf{P}^0 = [0,\, 1]$, corresponding to the channel at time $t = 0$ being in the bad state, we find

$$\begin{aligned} \mathbf{P}^m &= [0,\, 1]\mathbf{T}^m \\ &= [P^\infty(G)(1-(1-b-g)^m), \\ &\qquad P^\infty(B) + P^\infty(G)(1-b-g)^m]. \end{aligned} \tag{7}$$

When employing a forward error correcting code for a channel with memory, the performance might be poor in spite of a large error correcting capability. Because of the large degradation of the performance caused by the memory of the channel, a natural way to improve the performance is to use an interleaver in order to render the code symbols less dependent. For simplicity, let the interleaver be a block interleaver with m rows and n columns, where the bits that are to be transmitted are fed in row-wise and fed out column-wise (see Fig. 2).

How effective an interleaver is depends primarily on to what extent the interleaved symbols are dependent, which, in turn, depends on how large m, the *interleaving depth*, is chosen. The larger value of m, the better the interleaver can be expected to work, and if m is infinite, the performance would be the same as for a memoryless channel. In a practical situation, the size of the interleaver will typically be determined by how much delay can be accepted, and, with the size fixed, the dimensions are chosen to "match" the error correcting code. For the case of block coding, it is natural to choose n equal to the block length, since this is the smallest number of columns to avoid the "wrap-around" effect [9], and a larger value of n by necessity requires m to be decreased if the total size of the interleaver is fixed due to delay constraints. In light of (6) and (7), which explicitly give the probability that the channel has changed from one state to the other if observed m moments of time later, it is clear that the original GE channel and an interleaver will be equivalent to a GE channel where the corresponding transition probabilities b' and g', are [2]

$$b' = P^{\infty}(B)(1-(1-b-g)^m) \tag{8}$$

$$g' = P^{\infty}(G)(1-(1-b-g)^m). \tag{9}$$

From (8) and (9), the effect of the interleaving depth can be clearly seen. In particular, by letting $m \to \infty$, we obtain $b' = P^{\infty}(B)$ and $g' = P^{\infty}(G)$, as it would be for a memoryless channel.

While the (stationary) probability of being in the bad state is of relatively small interest, the probability distribution for being in the bad state d times out of n is critical when a block code of length n is used. The following theorem will therefore be useful.

Theorem 1: If the GE channel is observed at n consecutive instants of time, the probability that the channel is in the bad state d times, $0 \le d \le n$, is given by

$$P_n(d) = \begin{cases} P^{\infty}(G)(1-b)^{n-1}, & d = 0 \\ P^{\infty}(G)(P_n(d|GG) + P_n(d|GB)) \\ \quad + P^{\infty}(B)(P_n(d|BG) + P_n(d|BB)), & 1 \le d < n \\ P^{\infty}(B)(1-g)^{n-1}, & d = n \end{cases}$$

where

$$P_n(d|GG) = \sum_{i=2}^{\min(d+1,\, n-d)} \binom{n-d-1}{i-1}\binom{d-1}{i-2} \cdot (1-b)^{n-d-i} b^{i-1} (1-g)^{d-i+1} g^{i-1}$$

$$P_n(d|GB) = \sum_{i=1}^{\min(d,\, n-d)} \binom{n-d-1}{i-1}\binom{d-1}{i-1} \cdot (1-b)^{n-d-i} b^{i} (1-g)^{d-i} g^{i-1}$$

$$P_n(d|BG) = \sum_{i=1}^{\min(d,\, n-d)} \binom{n-d-1}{i-1}\binom{d-1}{i-1} \cdot (1-b)^{n-d-i} b^{i-1} (1-g)^{d-i} g^{i}$$

$$P_n(d|BB) = \sum_{i=2}^{\min(d,\, n-d+1)} \binom{n-d-1}{i-2}\binom{d-1}{i-1} \cdot (1-b)^{n-d-i+1} b^{i-1} (1-g)^{d-i} g^{i-1}.$$

Here $P_n(d|GG)$ is the conditional probability of being d times in the bad state, conditioned on being in the good state both the first and the last instants of time, and the other conditional probabilities are defined accordingly.

Proof: Theorem 1 is proved in the Appendix.

Using Theorem 1, the codeword error probability can be obtained exactly, and we summarize this result in the following theorem.

Theorem 2: If a t-error correcting block code of length n is used for error correction when the transmission is done over an interleaved GE channel, where the interleaving depth is m and the parameters in the GE channel are denoted in accordance to what have been used in this section, the probability of a codeword error is given by

$$P_{cw} = \sum_{d=0}^{n} P_n(d) \left[\sum_{e_b=0}^{d} \left(\binom{d}{e_b} P_e(B)^{e_b} (1-P_e(B))^{d-e_b} \cdot \sum_{e_g=\max(0,\, t+1-e_b)}^{n-d} \binom{n-d}{e_g} \cdot P_e(G)^{e_g} (1-P_e(G))^{n-d-e_g} \right) \right]$$

where $P_n(d)$ is found using Theorem 1 with the parameters b and g replaced by $b/(b+g)(1-(1-b-g)^m)$ and $g/(b+g)(1-(1-b-g)^m)$, respectively.

Proof: Assume that d symbols are received when the channel is in state B, and, consequently, $n-d$ symbols are received when the channel is in state G. Further, let e_b and e_g denote the number of symbols in error when the channel is in state B and G, respectively. Now, a codeword error occurs if $e_b + e_g > t$, and the probability of e_b and e_g errors in states B and G, respectively, conditioned on d, are given by [10, p. 417]

$$P(e_b|d) = \binom{d}{e_b} P_e(B)^{e_b} (1-P_e(B))^{d-e_b}$$

and

$$P(e_g|d) = \binom{n-d}{e_g} P_e(G)^{e_g} (1-P_e(G))^{n-d-e_g}.$$

It follows that the codeword error, conditioned on d, is given by

$$P_{cw}(d) = \sum_{e_b=0}^{d} \left(P(e_b|d) \sum_{e_g=\max(0,\, t+1-e_b)}^{n-d} P(e_g|d) \right).$$

This last expression simply says that if e_b symbol errors have occurred while the channel was in state B, at least $t+1-e_b$ errors must occur when the channel is in state G to cause a decoding error. Also, if $e_b > t$, i.e., the number of errors in state B is larger than the error correcting capability, the second sum is trivially equal to one. Finally, multiplying by $P_n(d)$ and summing over all d gives the desired result and completes the proof. □

III. MATCHING THE GILBERT–ELLIOTT CHANNEL TO THE LAND MOBILE CHANNEL

In order to use the results found in the previous section to evaluate the performance for a more realistic channel model, for which analytical results are not to be found, it is necessary to choose the parameters in the GE appropriately. In [3] and [5], a constructive way to match the GE channel model to a flat Rayleigh fading channel where the correlation function is given by the zeroth-order Bessel function, a model commonly used for the land mobile channel, e.g., [7] is given. The two channels are matched by choosing a level for the signal-to-noise ratio (SNR), where the channel is supposed to change state, and then match the average duration the fading amplitude is below this level to the average number of time units the GE channel is in the bad state. However, in [5] it is not verified that this gives accurate results. In [3], the model was verified by comparing the level-crossing rate. However, since this was the way in which the channels were matched, it is not surprising that the results are found to be in excellent agreement. Indeed, the choice of thresholds for verifying the model was arbitrary. The choice of threshold for the problem at hand, i.e., block-coded transmission, will affect the accuracy of the model, and for this reason the way the parameters of the GE model is found is described next.

First, recall that the amplitude α of the received signal is Rayleigh distributed, i.e.,

$$f(\alpha) = \frac{2\alpha}{\overline{\gamma}}\, e^{-\alpha^2/\overline{\gamma}}, \qquad \alpha \geq 0 \tag{10}$$

and the SNR is exponentially distributed, i.e.,

$$f(\gamma) = \frac{1}{\overline{\gamma}}\, e^{-\gamma/\overline{\gamma}} \qquad \gamma \geq 0 \tag{11}$$

where $\overline{\gamma}$ is the average SNR of the received signal. Since the GE channel consists of two states, we let α_t and γ_t be the thresholds for the amplitude and SNR observations, respectively, where the channel is considered to change state. We then calculate the stationary probabilities of finding the GE channel in its respective states by finding the fraction of time the Rayleigh fading channel is below and above γ_t, respectively, i.e.,

$$P^{\infty}(B) = \int_0^{\gamma_t} \frac{1}{\overline{\gamma}}\, e^{-\gamma/\overline{\gamma}}\, d\gamma = 1 - e^{-\gamma_t/\overline{\gamma}} = 1 - e^{-\rho^2} \tag{12}$$

where $\rho^2 = -\gamma_t/\overline{\gamma}$. Analogously, we have

$$P^{\infty}(G) = \int_{\gamma_t}^{\infty} \frac{1}{\overline{\gamma}}\, e^{-\gamma/\overline{\gamma}}\, d\gamma = e^{-\rho^2}. \tag{13}$$

Next, in order to match the dynamics of the two models, we consider the average time the amplitude remains below a certain level and match this to the average time the GE channel remains in the bad state once it has entered it. In doing this, we arrive at [5], [6]

$$g = \frac{\rho f_D T_s \sqrt{2\pi}}{e^{\rho^2} - 1} \tag{14}$$

$$b = \rho f_D T_s \sqrt{2\pi} \tag{15}$$

where $f_D = v/\lambda$ is the so-called Doppler frequency, and T_s is the symbol duration.

Finally, the error probabilities in the respective states in the GE channel are taken to be the conditional error probabilities of the Rayleigh fading channel, conditioned on being in the respective state, i.e.,

$$P_e(B) = \frac{1}{P^{\infty}(B)} \int_0^{\gamma_t} f(\gamma) P_e(\gamma)\, d\gamma \tag{16}$$

and

$$P_e(G) = \frac{1}{P^{\infty}(G)} \int_{\gamma_t}^{\infty} f(\gamma) P_e(\gamma)\, d\gamma. \tag{17}$$

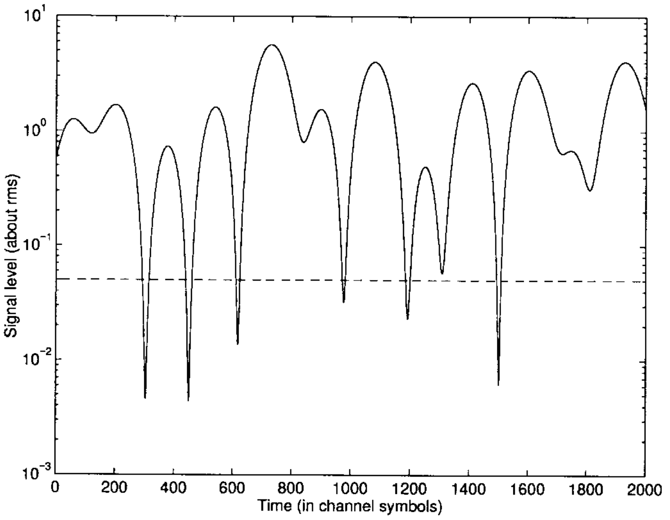

Fig. 3. Typical simulated Rayleigh fading when Jakes model is used (solid line) and a possible choice of level where the corresponding Gilbert–Elliott channel is going from one state to the other (dashed line).

Here, the factor in front of the integral equals the probability of being in the respective state, $f(\gamma)$ is given by (11), and $P_e(\gamma)$ is the symbol error probability for a given value of γ, which depends on the modulation format used.

In order to find how accurate one can expect the GE model to be when used to evaluate the performance of block-coded transmission over the land mobile channel, we need to specify some parameters for the latter channel. Toward that end, we will therefore make use of the following data if not stated otherwise:

- carrier frequency = 1.8 GHz;
- maximum delay due to the interleaver = 20 ms;
- information rate 9.6 kbit/s.

To see how the parameters are found, we go through one example in detail.

Example: Consider a situation where the modulation is BPSK, a double error correcting $(15, 7)$ binary BCH code is used for error correction and $\overline{\gamma} = 15$ dB. First, the dimensions of the interleaver are found. The maximum number of symbols in the interleaver equals $20 \cdot 9.6/r = 411$, where $r = 7/15$ is the rate of the code. Since the block length is 15, we choose this to be the number of columns in the interleaver. The interleaving depth will then be 27 code symbols $(15 \cdot 27 = 405)$. Next, one has to decide the threshold γ_t where the channel is considered to change state. In Fig. 3, a possible choice of γ_t, in this case 13 dB below the rms value, is shown together with the received SNR as a function of the time, in channel symbols, when the normalized Doppler frequency $f_D T_s$ equals 0.003 (corresponding to a vehicle speed of roughly 20 m/h). For this value of γ_t, $\rho^2 = 10^{-13/10} = 0.0501$, and we find $P^{\infty}(B) = 0.049$ and $P^{\infty}(G) = 0.951$.

For BPSK and a code of rate r, the conditional probability of a code symbol error, conditioned on the received SNR, is given by [10, p. 246] $P_e(\gamma) = Q(\sqrt{r2\gamma})$, where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt$$

TABLE I
THE PARAMETERS IN THE GILBERT–ELLIOTT MODEL FOR SOME DIFFERENT VALUES OF γ_t. $\overline{\gamma} = 15$ dB AND $f_D T_s = 0.003$

γ_t	$P^{\infty}(G)$	$P^{\infty}(B)$	$Pe(G)$	$Pe(B)$	g	b	P_{cw}
0 dB	0.969	0.031	8.11e-3	0.266	0.0416	1.34e-3	3.83e-3
1 dB	0.961	0.039	6.93e-3	0.243	0.0369	1.50e-3	4.38e-3
2 dB	0.951	0.049	5.70e-3	0.219	0.0328	1.68e-3	4.92e-3
3 dB	0.939	0.061	4.49e-3	0.195	0.0290	1.89e-3	5.40e-3
4 dB	0.924	0.076	3.35e-3	0.171	0.0256	2.12e-3	5.76e-3
5 dB	0.905	0.095	2.33e-3	0.147	0.0226	2.15e-3	5.93e-3
6 dB	0.882	0.118	1.49e-3	0.125	0.0199	2.67e-3	5.89e-3
7 dB	0.853	0.147	8.63e-4	0.105	0.0174	2.99e-3	5.66e-3
8 dB	0.819	0.181	4.37e-4	0.087	0.0152	3.36e-3	5.27e-3

is the complementary Gaussian distribution function. Therefore, referring to (16) and (17), it follows that

$$P_e(B) = \frac{1}{0.049}\int_0^{10^{-0.2}} 10^{-1.5} e^{-10^{-1.5}\gamma} Q\left(\sqrt{7/15 \cdot 2\gamma}\right) d\gamma = 0.219$$

and

$$P_e(G) = \frac{1}{0.951}\int_{10^{-0.2}}^{\infty} 10^{-1.5} e^{-10^{-1.5}\gamma} Q\left(\sqrt{7/15 \cdot 2\gamma}\right) d\gamma = 5.70 \cdot 10^{-3}.$$

Having found the error probabilities in each state, we proceed with the probabilities of going from one state to the other. Using (15), it follows that $\rho = -13$ dB $= 10^{-13/20} = 0.224$. Hence, using (14) and (15), we obtain

$$\begin{aligned} g &= \frac{\rho f_D T_s \sqrt{2\pi}}{e^{\rho^2} - 1} = 0.0328 \\ b &= \rho f_D T_s \sqrt{2\pi} = 1.68 \cdot 10^{-3}. \end{aligned}$$

Using these parameters in Theorem 2, one obtains that the codeword error probability equals $4.9 \cdot 10^{-3}$. □

Since one has to choose the parameter γ_t somewhat *ad hoc,* it is desirable that the probability of a codeword error is not dramatically changed by a minor variation of this parameter. That this is the case can be seen in Table I, where different values of the GE channel are presented for different values of γ_t. As seen from this table, although the channel parameters are sensitive to the choice of γ_t, the codeword error probability is not.

Also, by referring to Table I, we see that the performance for the matched GE model is poorest when γ_t is about 10 dB below $\overline{\gamma}$. The reason is that if γ_t is chosen either very small or very large, the channel behavior will be close to the memoryless case: If γ_t is chosen very small, the channel will seldom be in the bad state (b will be small relative to g), and the error probability in the good state will be close to the average error probability of the channel [see (13) and (17)]. Moreover, if the channel enters the bad state, it will not remain there for more than a few symbols (g will be large), so the interleaver will work properly. Alternatively, if γ_t is chosen very large, the channel will change states more often, again making the interleaving more effective. This is due to the fact that the average time between two successive changes of the channel state equals $(\overline{T}(G) + \overline{T}(B))/2$, which, in the region of interest, is roughly $1/(2b)$. Also, with γ_t very large, the error probability in the bad state will not be that high. Since this means that the difference between the two states is diminished, the effect of remaining in the bad state will not be that severe and, consequently, the performance will be improved. In effect, this says that the exact choice of γ_t is not critical for the final result. In particular, for the results presented in the next section, where the GE channel is used to compare codes of different complexity, the value of γ_t that was chosen corresponds to the worst-case probability of error.

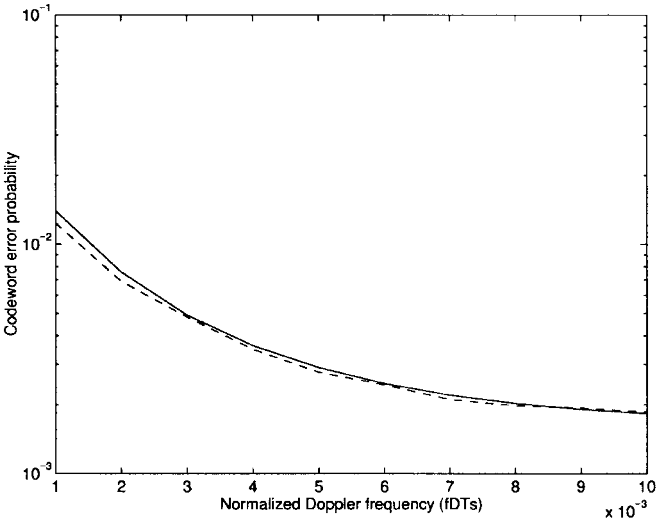

Fig. 4. Codeword error probability as a function of the normalized fading rate. Numerically evaluated when the Gilbert–Elliott model is used with $\gamma_t = 2$ dB (solid line) and simulated (dashed line).

Since the interleaving works better the faster the mobile is moving, it is interesting to see how the performance is affected by increasing the Doppler frequency. Fig. 4 shows the codeword error probability as a function of the normalized Doppler frequency when the GE model is used for $\gamma_t = 2$ dB, i.e., 13 dB below the rms value, together with the corresponding result obtained by simulation. The simulation results are obtained by using the model proposed by [11] and described in detail in [12, pp. 181–184]. The fading amplitude is assumed to remain constant during the entire symbol, so that once the fading amplitude is known, the probability of a symbol error can be computed analytically.

As a perspective, if the channel was assumed to be memoryless, the resulting codeword error is 0.0016, which for low Doppler frequencies cannot be considered a good approximation.

In Fig. 5, the codeword error probability as a function of the received SNR is presented when a $(15, 7)$ binary BCH code is used and $f_D T_s = 0.003$. The solid lines are analytically derived, and the dashed lines are obtained by simulations. The upper pair of curves corresponds to a delay of 10 ms, and the lower pair to an interleaver delay of 20 ms. The dash-dotted curve corresponds to a memoryless channel. As seen from Fig. 5, results obtained by using the Gilbert–Elliott model are very accurate, while assuming a memoryless channel will give results that are too optimistic by about an order of magnitude.

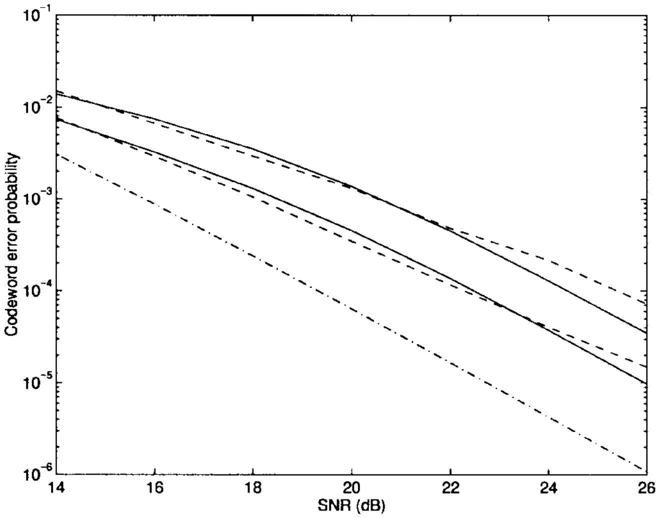

Fig. 5. Codeword error probability as a function of the received SNR. Numerically evaluated when the Gilbert–Elliott model is used, for $\gamma_t = 13$ dB below E_b/N_0 (solid lines) and simulated (dashed line). Memoryless channel (dash–dotted line). See text for further information.

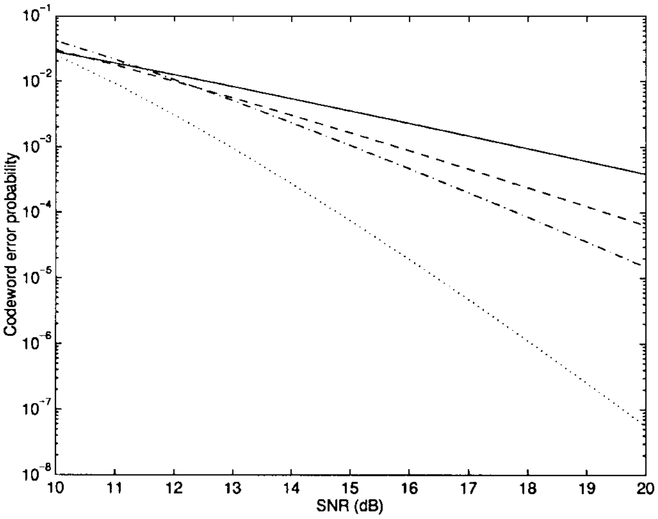

Fig. 6. Codeword error probability as a function of the received SNR for the following binary BCH codes; (7, 4) (solid line), (15, 7) (dashed line), (31, 16) (dash–dotted line), and (63, 30), (dotted line). Memoryless channel.

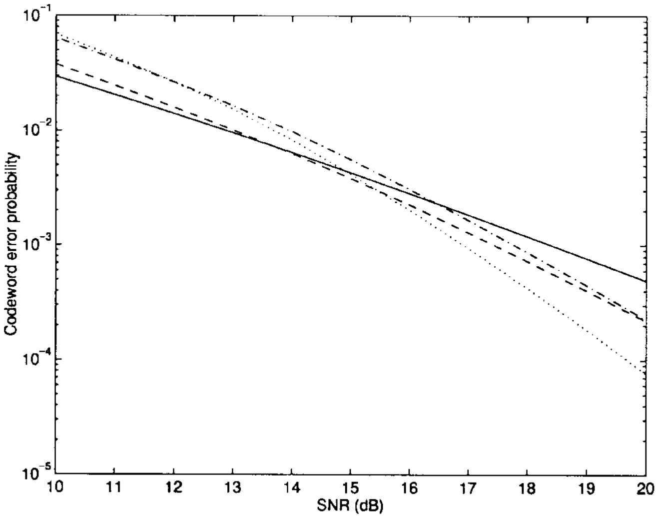

Fig. 7. Codeword error probability as a function of the received SNR for the following binary BCH codes; (7, 4) (solid line), (15, 7) (dashed line), (31, 16) (dash–dotted line), and (63, 30), (dotted line). $f_D T_b = 0.010$.

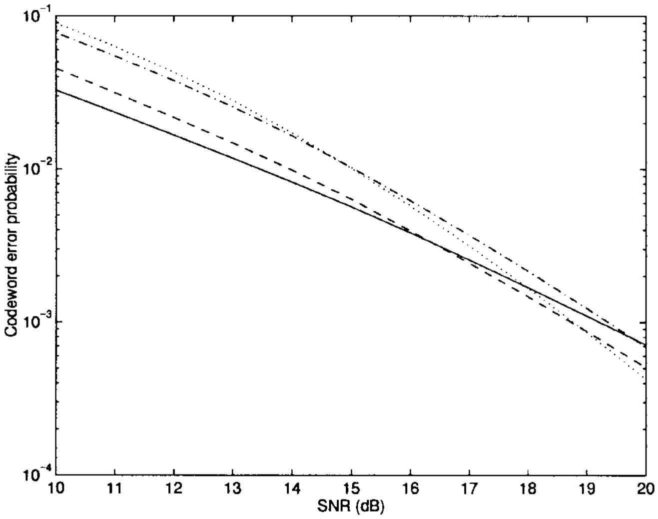

Fig. 8. Codeword error probability as a function of the received SNR for the following binary BCH codes; (7, 4) (solid line), (15, 7) (dashed line), (31, 16) (dash–dotted line), and (63, 30), (dotted line). $f_D T_b = 0.006$.

IV. COMPARISONS OF SOME BLOCK CODES

Having seen that the Gilbert–Elliott model can be used to, in an accurate way, estimate the codeword error probability for block-coded transmission over the land mobile channel, we will now use the former in order to evaluate the effect that the choice of error correcting code and interleaver has on the performance. Since the interleaver, or more precisely, the interleaving depth, will be of great importance in order for the code to work appropriately, it is not clear which code will give the best result. This is because using a more powerful code typically implies that the block length is increased, and therefore the interleaving depth has to be reduced, in order to meet the delay constraint. Consequently, the interleaver will work better for a code with smaller block length.

In the figures below, the performance is compared for the following binary BCH codes of approximately the same rate: (7, 4), (15, 7), (31, 16), and (63, 30). The error correcting capabilities for these codes are 1, 2, 3, and 6, respectively. First, in Fig. 6, the four codes are compared for a memoryless channel (perfect interleaving). The improvement in performance by using a more powerful code is clearly seen here. Then, in Figs. 7 and 8, the same codes are compared for situations where the interleaving is not perfect due to a delay constraint of 20 ms. The Doppler frequency is in these figures is normalized by the bit duration, rather than the duration of a code symbol, since the latter depends on the code used while the former does not. As seen from Figs. 7 and 8, the outstanding performance of the more powerful code is greatly degraded, and, in some cases, is even worse than the performance of the simpler codes.

Finally, in Fig. 9, the codeword error is depicted as a function of the normalized Doppler frequency, for SNR = 16 dB. As expected, for large values of the Doppler frequency the

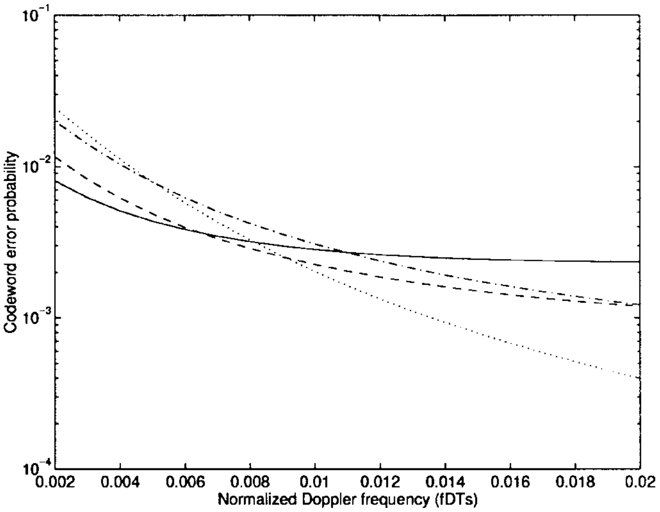

Fig. 9. Codeword error probability as a function of the normalized Doppler frequency for the following binary BCH codes; (7, 4) (solid line), (15, 7) (dashed line), (31, 16) (dash–dotted line), and (63, 30), (dotted line). SNR = 16 dB.

interleaver works properly, resulting in the most powerful code performing significantly the best, whereas for small values of the Doppler frequency, the situation essentially reverses.

V. CONCLUSION

In this paper, block-coded transmission over the land mobile channel was analyzed by using the well-known Gilbert–Elliott model. The accuracy of using the Gilbert–Elliott model was demonstrated by comparing analytical results for the Gilbert–Elliott channel with simulation results of the land mobile channel, showing both that the model was accurate for a broad class of parameters, e.g., delay constraint, speed of the mobile, and SNR, and that moderate variations of the parameters in the Gilbert–Elliott model only caused small variations in the codeword error probability. Finally, the Gilbert–Elliott model was used to compare the performance of different codes to see the effect of imperfect interleaving. It was found that, as the block length was increased, which corresponds to using a more powerful code, the interleaving depth had to be reduced, and, because of the latter, using a code of relatively small block length was as good as, and sometimes better than, using a long block length code.

APPENDIX

Recall that we want to find the probability of being in state B exactly d out of n times. For the cases $d = 0$ and $d = n$, the result is trivial, since $d = 0$ means that the channel starts in the good state and never leaves it, which will happen with probability $P^{\infty}(G)(1-b)^{n-1}$, and $d = n$ means that the channel starts in the bad state and remains there, which will happen with probability $P^{\infty}(B)(1-g)^{n-1}$. Henceforth, we may therefore assume that $1 \le d < n$.

Now, consider $P_n(d|GG)$, and assume that the channel behavior depicted below resulted in the channel being in the bad state exactly d times.

- The first g_1 instants of time the channel is in the good state.
- The next b_1 instants of time the channel is in the bad state.
- $\vdots$
- The next b_{i-1} instants of time the channel is in the bad state.
- The last g_i instants of time the channel is in the good state.

i is the number of sojourns in the good state, and $2(i-1)$ is the number of times the channel changes state. The probability for this specific channel behavior, conditioned on the initial state, is given by

$$\begin{aligned}(1-b)^{g_1-1}b(1-g)^{b_1-1}g\cdots(1-g)^{b_{i-1}-1}g(1-b)^{g_i-1}\\ = (1-b)^{n-d-i}b^{i-1}(1-g)^{d-i+1}g^{i-1} \qquad (18)\end{aligned}$$

where we have used the facts that

$$\sum_{k=1}^{i} g_k = n-d \qquad (19)$$

and

$$\sum_{k=1}^{i-1} b_k = d. \qquad (20)$$

From this expression, we have that the probability, apart from n and d, only depends on the number of times the channel state changes, not on the exact behavior of the channel. We also note that $i \le d+1$, otherwise the channel would be in the bad state too many times, and $i \le n-d$, otherwise the channel would be in the good state too many times.

Now, since the number of ways d can be expressed as a sum of $i-1$ positive integers is [14, p. 149]

$$\binom{d-1}{i-2} \qquad (21)$$

and the number of ways that $n-d$ can be expressed as a sum of i positive integers is

$$\binom{n-d-1}{i-1} \qquad (22)$$

we have that

$$\begin{aligned}P_n(d|GG) = \sum_{i=2}^{\min(d+1,n-d)} \binom{d-1}{i-2}\binom{n-d-1}{i-1}\\ \cdot(1-b)^{n-d-i}b^{i-1}(1-g)^{d-i+1}g^{i-1}. \qquad (23)\end{aligned}$$

By using similar arguments, it is straightforward to derive the other conditional probabilities. Finally, $P_n(d)$ is found by summing the conditional probabilities, weighted appropriately, i.e.,

$$\begin{aligned}P_n(d) = P^{\infty}(G)(P_n(d|GG) + P_n(d|GB))\\ + P^{\infty}(B)(P_n(d|BG) + P_n(d|BB)) \qquad (24)\end{aligned}$$

which concludes the proof.

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Leif Wilhelmsson (S’92–M’98) was born in Emmaboda, Sweden, in 1967. He received the M.S. degree in electrical engineering in 1992 and the Ph.D. degree in telecommunication theory in 1998, both from Lund University, Sweden.

Since March 1998, he has been with R&D New Systems at Ericsson Mobile Communications AB, Lund. His present research interests are in error control coding and modulation for spread spectrum communications.

Laurence B. Milstein (M’68–SM’77–F’85), for a photograph and biography see p. 88 of the January 1999 issue of this TRANSACTIONS